Physics 209
Fall 2002
Homework 8
Due Friday, October 18 at 5:00pm.

Reading Assignment: Read the notes on Thomas Precession, and proceed into Chapter 12 of Jackson to keep up with lectures. I have also posted some optional notes on the Lie algebra of the Lorentz group, that several students were interested in. I did not lecture on this in class and will not expect you to know it. Jackson has a section on this subject that he uses in discussing Thomas precession, but since I took another approach that does not use the Lie algebra, I will skip this topic.

1. This is a variation on Jackson's problem 12.11. Charged leptons (the electron, muon and tau) possess a magnetic moment with a g factor very close to 2. The small difference, here measured by the quantity

$$a = \frac{g}{2} - 1, \tag{1}$$

can be calculated to high precision theoretically. Recently new measurements of the muon g factor have caused excitement because they seem to disagree with theory based on the standard model, thereby revealing physics beyond the standard model. The vaues given by both theory and experiment have error bars, in the case of theory because the theoretical calculations must rely on extrapolations from other experimental data for the effects of the strong interaction on the g factor. The error bars are currently not as small as one would like.

Nonrelativistically, the orbital frequency of a charged particle with $g = 2$ in a uniform magnetic field is the same as the spin precession frequency. You can easily verify this for yourself. Thus, if $g = 2$ exactly, the spin undergoes one cycle of precession when the particle returns to some initial point on its circular orbit in the magnetic field. If g differs from 2 by a small amount, then the spin acquires some small net precession angle after a single orbital period. Experiments take advantage of this fact to measure $g - 2$ to high precision. Real experiments work with relativistic particles, however.

(a) In class we derived the BMT equation, Jackson (11.164),

$$\frac{ds^\mu}{d\tau} = \frac{e}{mc}\left[\frac{g}{2}F^\mu{}_\nu s^\nu - \frac{1}{c^2}\left(\frac{g}{2}-1\right)(u \cdot F \cdot s)u^\mu\right], \tag{2}$$

where e is the signed charge of the particle and $(u\cdot F\cdot s) = u^\alpha\, F_{\alpha\beta}\, s^\beta$. This equation is based on the assumption that the evolution of the spin is a combination of Fermi-Walker transport with a precession induced by the magnetic field in the rest frame, plus the assumption that the acceleration of the particle is produced by electromagnetic forces.

Assuming that s^μ is purely spatial in a rest frame of the particle at some initial time ($(s\cdot u) = 0$), and assuming that the spin evolves according to the BMT equation, show that s^μ remains a purely spatial vector in a rest frame of the particle for all times. Next show that if q^μ, p^μ are two vectors that are purely spatial in any rest frame of the particle and that evolve according to the BMT equation, then $(p\cdot q)$ is constant.

(b) In the lab frame a muon is orbiting in a circular orbit in a uniform magnetic field $\mathbf{B}$, as in a storage ring. Let s^i be the components of the spin in the Thomas conventional rest frame with basis vectors $\{e_\alpha\}$, as in the notes on Thomas precession. Note that $s^0 = 0$. Find an equation for $ds^i/d\tau$ in terms of the magnetic field $\mathbf{B}$ in the lab frame. You can write the vector with components s^i as $\mathbf{s}$, but remember this vector is measured in the Thomas conventional rest frame.

(c) Find the net angle of precession as seen in the Thomas conventional rest frame after one orbital period. Express your answer in terms of a.

2. In class we discussed how the inertia of an object depends on its energy content. This is part of understanding the relation $E = mc^2$ between energy and mass. As was explained in class, we may define the inertial mass of an object by using $F = ma$, applied when the velocity is very small (or zero). The object itself may contain subsystems with large velocities, however, such as the photons bouncing in the box of mirrors we discussed in class. For another example, the electrons in orbit around the atoms of some object have kinetic energy, and this kinetic energy contributes to the mass. So does the potential energy of the electrons. Since the sum of the kinetic and potential energies is negative for a bound electron, an atom has less inertial mass than the free electrons and nuclei out of which it is composed.

Suppose our "atom" is a charged particle in a circular orbit in a uniform magnetic field, that is, with zero velocity along the direction of the magnetic field. Apply a weak electric field to this system in the direction parallel to the magnetic field, and compute the small velocity in this direction produced after a short time. Show that this gives an inertial mass that includes the kinetic energy of the circular motion of the charged particle.

3. A particle of charge e finds itself in the vacuum light wave,

$$\mathbf{E} = E_0 \hat{\mathbf{x}} \cos\phi,$$
$$\mathbf{B} = E_0 \hat{\mathbf{y}} \cos\phi, \tag{3}$$

where $\phi = \omega t - kz$. The initial conditions are $\mathbf{x} = 0$ and $\mathbf{v} = 0$ at $t = 0$. Find the motion of the particle. Show that averaged over a long time, the particle moves in the z direction with a velocity,

$$c \frac{\omega_0^2}{\omega_0^2 + 4\omega^2}, \tag{4}$$

where $\omega_0 = eE_0/mc$. Hint: First show that the phase of the wave, as seen by the particle, is proportional to the particle's proper time. This is a problem for which the nonrelativistic equations are harder to solve than the relativisitic ones.

4. Hamilton's principle says only that the action is stationary along physically realizable motions, not that it is minimum or extremum (in spite of what many books say). Nevertheless, it is interesting to investigate when the action actually is minimum. This is also a good way to make sure you really understand Hamilton's principle. Consider the one dimensional harmonic oscillator with Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2, \tag{5}$$

and consider a physically realizable orbit which begins at $(x_0, t = 0)$ and which ends at $(x_1, t = T)$. Show that if $T < \pi/\omega$, then the action actually is a minimum. Remember that the variations $\delta x(t) = \epsilon(t)$ you consider must vanish at the endpoints, $\epsilon(0) = \epsilon(T) = 0$. Show that for an arbitrary time T, the action is generally at a saddle point in function space when evaluated on a physically realizable orbit, and that there are an infinity of directions in which it is increasing, and a finite number in which it is decreasing. Show that the number in which it is decreasing is largest integer less than $\omega T/\pi$.

Hint: Evaluate the action exactly, don't just expand it out to first order in ϵ. Since $\epsilon(0) = \epsilon(T) = 0$, use the complete set of orthothogonal functions $\sin(n\pi t/T)$ as a basis, to expand $\epsilon(t)$ on the interval $(0, T)$.