Physics 209
Fall 2002
Homework 6
Due Friday, October 11 at 5:00pm.

Reading Assignment: Read the notes on Tensor Analysis.

1. This is basically Jackson's problem 11.5, but do it this way. Let the primed frame be moving with velocity $\mathbf{v}_0$ with respect to the unprimed (lab) frame. Let $\beta_0 = v_0/c$ and $\gamma_0 = 1/\sqrt{1-\beta_0^2}$. Let a particle have velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ in the lab frame, and $\mathbf{v}'$ and $\mathbf{a}'$ in the moving frame. Find an expression for $\mathbf{v}'$ and $\mathbf{a}'$ in terms of $\mathbf{v}$ and $\mathbf{a}$.

2. In class we derived the expressions for a boost (a Lorentz transformation) along the x-axis, starting from Einstein's postulates. Then we claimed that this Lorentz transformation preserves the fundamental invariant,

$$c^2t^2 - |\mathbf{x}|^2 = g_{\mu\nu}\, x^\mu x^\nu. \tag{1}$$

We showed that the expression (1) is invariant under a Lorentz transformation,

$$x'^\mu = L^\mu{}_\nu\, x^\nu, \tag{2}$$

if the matrix L of the Lorentz transformation satisfies

$$L^t g L = g, \tag{3}$$

where L has components $L^\mu{}_\nu$ (μ = row, ν = col), and g has components $g_{\mu\nu}$. Verify that the boost along the x-axis does satisfy Eq. (3).

If frames K and K' are related by a Lorentz transformation, and frames K' and K'' are also related by a Lorentz transformation, then frames K and K'' must be related by a Lorentz transformation. This means that the product of two Lorentz transformations must be a Lorentz transformation (property 1). The inverse of a Lorentz transformation must also be a Lorentz transformation (property 2). Thus, the set of all Lorentz transformations forms a group. Verify that the set of matrices L that satisfies Eq. (3) also satisfies properties 1 and 2.

Show that a spatial rotation (a rotation of the space coordinates that does nothing to the time coordinate) is a Lorentz transformation. General Lorentz transformations can be built up as products of boosts along the three axes and of rotations about the three axes. For proper Lorentz transformations, the rotations must be proper. Improper Lorentz transformations are generated if parity and time-reversal are included in the mix.

Show that the component $L^0{}_0$ of a Lorentz transformation satisfies

$$|L^0{}_0| \geq 1. \tag{4}$$

Lorentz transformations for which $L^0{}_0 < 0$ are those which reverse the direction of time (they are improper).

3. Let frame K' be moving in the x-direction with velocity v_1 with respect to frame K, let frame K'' be moving in the x-direction with velocity v_2 with respect to frame K', and let v be the velocity of frame K'' with respect to frame K. Show that if $|v_1| < c$ and $|v_2| < c$, then $|v| < c$.

4. Some problems on tensor analysis.

(a) Given that g_{ij} (the metric tensor) is a tensor, and that g^{ij} is defined as the inverse matrix, show that g^{ij} is a tensor. See Eq. (4.19) in the notes on Tensor Analysis.

(b) If the Jacobian matrix is constant (for linear coordinate transformations), show that $\partial_i A^j$ is a mixed tensor if A^j is a contravariant vector. Show that this is not in general true if the Jacobian matrix is not constant.

(c) Let g_{ij} be a metric tensor in the plane, on which we use rectilinear coordinates and only consider linear coordinate transformations. Then g_{ij} is a constant matrix (in a given coordinate system, the components of g_{ij} do not depend on x^i). Show that if g_{ij} is positive definite, then there exists a coordinate system in which $g_{ij} = \delta_{ij}$. Show that if g_{ij} has one positive and one negative eigenvalue in some coordinate system, then there exists a coordinate system in which the metric has the form,

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

(d) Consider skew axes (not orthogonal) in the Euclidean plane, specified by unit vectors $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\eta}}$. Let coordinates ξ and η be measured by distance along the two axes. Also write $\xi = x^1$, $\eta = x^2$. Let $\mathbf{V}$ be a vector with base at the origin and tip at some point P in the plane. Let (V^1, V^2) be the contravariant components of $\mathbf{V}$. Show that V^1 is obtained by projecting P onto the ξ-axis, moving parallel to the η-axis, and similarly for V^2. Show that the covariant components V_1, V_2 are obtained by projecting P perpendicularly to the axes.

(e) Show that if a purely covariant or purely contravariant tensor, say, A_{ij} or B^{ij}, is either symmetric (for example, $B^{ij} = B^{ji}$) or antisymmetric (for example, $A_{ij} = -A_{ji}$) in one coordinate system, then this property holds in all coordinate systems. Consider general coordinate transformations (not just linear ones). Show, however, that the same is not generally true for a mixed tensor (the Kronecker δ^i_j is an exception).

5. Write out the homogeneous Maxwell equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -(1/c)\partial \mathbf{B}/\partial t$ in components, to make four equations. Transform the derivatives in x, y, z, t to primed variables, using the boost with velocity v down the x-axis. Show that the new equations can be written as a primed version of the original Maxwell equations only if $\mathbf{E}'$ and $\mathbf{B}'$ are certain functions of $\mathbf{E}$, $\mathbf{B}$ and v which are uniquely determined, apart from an overall multiplicative factor A. As was discussed in class, this factor can be shown to be equal to 1 by using the inverse transformation and arguments of isotropy of space.