Physics 209

Fall 2002

Homework 4

Due Friday September 27 at 5:00pm.

Reading Assignment: To keep up with the lectures during the week of September 23-27, read Chapter 6 through section 6.6, then start on Chapter 11.

1. The primed frame is moving with velocity $\mathbf{v}$ relative to the unprimed frame (the lab frame). Using the nonrelativistic transformation laws (a Galilean transformation) and the definitions of the electric and magnetic fields given in the notes on units, find $\mathbf{E}'$ and $\mathbf{B}'$ in terms of $\mathbf{E}$ and $\mathbf{B}$. This is easy, but worth thinking about.

2. In class we showed that if $C(t)$ is a moving loop, then

$$\oint_{C(t)} (\mathbf{E} + \mathbf{v}\times\mathbf{B}) \cdot d\boldsymbol{\ell} = -\frac{dF}{dt}, \tag{1}$$

where $\mathbf{v}$ is the velocity of the loop (allowed to depend upon position on the loop), where F is the flux through the loop and the time derivative is the total time derivative, taking into account any possible time dependence of $\mathbf{B}$ as well as the motion of the loop. This follows by applying some simple mathematics to Faraday's equation,

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{2}$$

The problem is to interpret Eq. (1) physically.

Assume the curve $C(t)$ coincides with a wire. We will adopt the following model for what is going on inside the wire. First, we assume the wire is very thin. Next, consider a segment of the wire of length $d\ell$, or $d\boldsymbol{\ell} = \hat{\boldsymbol{\ell}}\, d\ell$ as a vector. The sums in the following refer to the charges inside the wire segment $d\ell$. There are two kinds of charges, electrons (q_i) and ions (Q_i). Electron q_i moves with velocity $\mathbf{u}_i$ relative to the wire, that is, with velocity $\mathbf{v}+\mathbf{u}_i$ relative to the lab frame. (We use nonrelativistic mechanics in this problem.) Each electron is acted upon by four forces, the electric force $\mathbf{F}_{Ei} = q_i\mathbf{E}$, the magnetic force $\mathbf{F}_{Bi} = q_i(\mathbf{v}+\mathbf{u}_i)\times\mathbf{B}$, the constraining force of the wire $\mathbf{F}_{wi}$, and the force of resistance $\mathbf{F}_{Ri}$. The constraining force exerted by the wire itself is perpendicular to the wire, $\mathbf{F}_{wi} \cdot \hat{\boldsymbol{\ell}} = 0$, while the force of resistance is parallel to the wire, $\mathbf{F}_{Ri}\times\hat{\boldsymbol{\ell}} = 0$. Think of the force of resistance as being like the frictional force you get when pulling marbles through mollasses.

The elctron mass is very small, so we ignore the term $m\mathbf{a}$ in $\mathbf{F} = m\mathbf{a}$. The physical picture is that if there is any unbalanced force on the electron, it instantly accelerates parallel to the wire to acquire a new velocity $\mathbf{u}_i$ to make the frictional force cancel out the unbalanced force parallel to the wire, and the force perpendicular to the wire is cancelled out by the force of constraint supplied by the wire itself. Thus, the total force on the electron vanishes,

$$\mathbf{F}_{Ei} + \mathbf{F}_{Bi} + \mathbf{F}_{wi} + \mathbf{F}_{Ri} = 0. \tag{3}$$

The electrons are like massless beads on a wire that can slide along the wire (but with friction).

As for the ions Q_i, we assume that they do not move relative to the wire, so their velocity is $\mathbf{v}$. We will also assume for simplicity that $\dot{\mathbf{v}} = 0$, so we do not have to worry about the kinetic energy of the wire in the energy balance. Thus the total force on the ions vanishes,

$$\mathbf{F}_{Ei} + \mathbf{F}_{Bi} + \mathbf{F}_{wi} = 0, \tag{4}$$

where now $\mathbf{F}_{Ei} = Q_i\mathbf{E}$ and $\mathbf{F}_{Bi} = Q_i\mathbf{v}\times\mathbf{B}$ and $\mathbf{F}_{wi}$ is the mechanical force on the ion exerted by the wire itself. For the ions, this force does not have to be perpendicular to the wire. Ultimately, the mechanical forces on the electrons and ions in the wire are supplied by whoever is moving the wire through the magnetic field.

Finally, we assume the wire is neutral,

$$\sum_i q_i + \sum_i Q_i = 0. \tag{5}$$

Also, the current in the wire is given by

$$I\,d\boldsymbol{\ell} = \sum_i q_i\mathbf{u}_i. \tag{6}$$

Under these assumptions, let us compute the rate at which electric forces do work on the charges in the wire. This is

$$\sum_i q_i\mathbf{E}\cdot(\mathbf{v}+\mathbf{u}_i) + \sum_i Q_i\mathbf{E}\cdot\mathbf{v} = \sum_i q_i\mathbf{E}\cdot\mathbf{u}_i = I\mathbf{E}\cdot d\boldsymbol{\ell}, \tag{7}$$

where we use Eqs. (5) and (6). Thus, the total rate at which electric forces do work on the charges in the wire is given by

$$\oint_{C(t)} I\mathbf{E}\cdot d\boldsymbol{\ell}. \tag{8}$$

If I is uniform around the wire, then it can be taken out of the integral and we obtain I times the first term of Eq. (1).

(a) Compute the rate at which magnetic forces do work on the charges in the wire.

(b) Compute the rate at which mechanical forces or constraint forces do work on the charges in the wire.

3. Jackson, problem 5.14.

4. This problem is inspired by Jackson problem 5.22. This problem concerns magnetostatics, for which $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$. An infinitely permeable medium is an idealization for which some explanation may help. In a linear medium, $\mathbf{B} = \mu\mathbf{H}$, so if $\mu \to \infty$, then either $\mathbf{B} \to \infty$ or $\mathbf{H} \to 0$. Let us choose the latter, so $\mathbf{B}$ is finite. Then assuming there are no free currents at interfaces, when we cross from an infinitely permeable medium into vacuum, boundary conditions say that $\mathbf{H}$ outside must be normal to the surface, and hence so must $\mathbf{B}$.

(a) Let the region $z < 0$ be filled with an infinitely permeable medium, and let there be a given current distribution $\mathbf{J}(\mathbf{x})$ in the region $z > 0$. Use the uniqueness theorem for the Poisson equation to show that there is a unique magnetic field $\mathbf{B}(\mathbf{x})$ in the region $z > 0$ that satisfies $B_x = B_y = 0$ at $z = 0$ and the magnetostatic field equations for $z > 0$. Thus, if any solution to those field equations and boundary conditions can be found by any means (for example, guessing), it must be the right answer.

Show that the desired solution in the region $z > 0$ can be represented as the magnetic field produced by the given current distribution $\mathbf{J}(\mathbf{x})$ in the region $z > 0$, plus the field produced by an image current distribution $\mathbf{J}_i(\mathbf{x})$ in the region $z < 0$, but without the infinitely permeable medium. Find the relation between $\mathbf{J}_i(\mathbf{x})$ for $z < 0$ and $\mathbf{J}(\mathbf{x})$ for $z > 0$, that is, write $\mathbf{J}_i(x, y, -z)$ in terms of $\mathbf{J}(x, y, z)$.

So we have the solution in the region $z > 0$. With the medium restored in the region $z < 0$, the magnetic field in the region $z > 0$ must be the sum of the field produced by $\mathbf{J}(\mathbf{x})$ and that produced by some bound current distribution in the magnetic medium (since there are no other currents around). As we shall see, there are no bound volume currents, only surface currents. Thus, the field $\mathbf{B}$ in the region $z > 0$ satisfying the desired boundary conditions is the sum of the field produced by $\mathbf{J}(\mathbf{x})$ and that produced by some surface current distribution $\mathbf{K}(\mathbf{x})$ on $z = 0$. Write $\mathbf{B} = \mathbf{B}^J + \mathbf{B}^K$ for the sum of these two magnetic fields. Thus, in the region $z > 0$, the field $\mathbf{B}^K$ is the same as the field produced by the image current. This will not be true in the region $z < 0$, in fact so far we do not know what $\mathbf{B}$ is in the region $z < 0$.

(b) Suppose the permeability μ in the region $z < 0$ is not infinite (but the medium is still linear). Assuming there are no free currents in the region $z < 0$, explain why there are no bound currents either, at least no bound volume currents. Thus, the only bound currents possible are on the surface. We may assume this state of affairs persists in the limit $\mu \to \infty$. Henceforth we go back to the case $\mu = \infty$.

(c) Without making any assumptions about the current distribution $\mathbf{K}$ in the plane $z = 0$, find a relation between $\mathbf{B}^K(x, y, z)$ and $\mathbf{B}^K(x, y, -z)$. Then show that

$$\mathbf{K} = \frac{2}{\mu_0}\hat{\mathbf{z}}\times\mathbf{B}^J, \tag{9}$$

where $\mathbf{B}^J$ is evaluated at $z = 0$. Thus, assuming we can find the magnetic field produced by $\mathbf{J}$, we know the surface current.

(d) Now find a relation between $\mathbf{B}^i(x, y, z)$ and $\mathbf{B}^J(x, y, z)$, where $\mathbf{B}^i$ is the field produced by the image current, and then use this to find a simple expression for the total magnetic field in the region $z < 0$ in the presence of the infinitely permeable medium.

(e) Finally, find an expression for the z-component of the force which the magnetic field produced by $\mathbf{J}$ exerts on the surface current $\mathbf{K}$. Recall that when finding the force exerted by a magnetic field on a surface current, you must use the average of the field on the two sides. Express your answer in terms of $\mathbf{B}^J$ evaluated on the surface. This expression can be used to find the force with which a magnet sticks to an iron surface, as in Jackson's problem.

5. This is basically Jackson's problem 6.7, but I've tried to simplify the notation a little and give you a few more hints. Also note Jackson's hints.

The following is the notation used in lecture. Let q_{fj}, $\mathbf{x}_{fj}$ and $\mathbf{v}_{fj}$ be the charge, position, and velocity of the j-th free charge. Let n be the index used to label molecules. Let q_n, $\mathbf{x}_n$ and $\mathbf{v}_n$ be the charge, center of mass position, and center of mass velocity of the n-th molecule. Let q_{nj}, $\mathbf{x}_{nj}$ and $\mathbf{v}_{nj}$ be the charge, position, and velocity, the latter two with respect to the center of mass of molecule n, of the j-th charge in molecule n. Thus, the position of the j-th particle in molecule n is $\mathbf{x}_n + \mathbf{x}_{nj}$.

Define the total charge, electric dipole moment, second moment tensor, and magnetic moment of the n-th molecule by

$$q_n = \sum_j q_{nj}, \tag{10}$$

$$\mathbf{p}_n = \sum_j q_{nj}\, \mathbf{x}_{nj}, \tag{11}$$

$$t_n^{\alpha\beta} = \sum_j q_{nj}\, x_{nj}^{\alpha} x_{nj}^{\beta} \tag{12}$$

$$\mathbf{m}_n = \tfrac{1}{2} \sum_j q_{nj}\, \mathbf{x}_{nj} \times \mathbf{v}_{nj}, \tag{13}$$

where $\alpha, \beta = 1, 2, 3$ for x, y, z indices, where I have not attempted to express $t_n^{\alpha\beta}$ in terms of the standard quadrupole moment tensor as Jackson has done, and where the superscript or subscript position of the indices α, β has no significance.

Now we define some macroscopic, averaged fields. In the following, f is the window function of mesoscopic scale, as discussed in class. First the free charge density:

$$\rho_f(\mathbf{x}) = \sum_j q_{fj}\, f(\mathbf{x} - \mathbf{x}_{fj}) = \left\langle \sum_j q_{fj}\, \delta(\mathbf{x} - \mathbf{x}_{fj}) \right\rangle. \tag{14}$$

Next the molecular (or ionic) charge density:

$$\rho_m(\mathbf{x}) = \sum_n q_n\, f(\mathbf{x} - \mathbf{x}_n) = \left\langle \sum_n q_n\, \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle. \tag{15}$$

Next the electric polarization:

$$\mathbf{P}(\mathbf{x}) = \sum_n \mathbf{p}_n\, f(\mathbf{x} - \mathbf{x}_n) = \left\langle \sum_n \mathbf{p}_n\, \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle. \tag{16}$$

Next the second moment tensor,

$$T_{\alpha\beta} = \sum_n t_n^{\alpha\beta}\, f(\mathbf{x} - \mathbf{x}_n) = \left\langle \sum_n t_n^{\alpha\beta}\, \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle, \tag{17}$$

which is used to define the vector $\mathbf{R}$ by

$$R_\alpha = \frac{\partial T_{\alpha\beta}}{\partial x_\beta}. \tag{18}$$

Then in class we showed that the averaged charge density is

$$\langle \eta \rangle = \rho_f + \rho_m - \nabla \cdot \mathbf{P} + \tfrac{1}{2} \nabla \cdot \mathbf{R} + \ldots, \tag{19}$$

which motivates the definition,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} - \tfrac{1}{2} \mathbf{R} + \ldots, \tag{20}$$

so that the Maxwell equation becomes

$$\nabla \cdot \mathbf{D} = \rho_f + \rho_m. \tag{21}$$

The $\mathbf{R}$ or $T_{\alpha\beta}$ term is negligable in comparision to the $\mathbf{P}$ term, and I wouldn't even bother you with it except that it is needed in the averaging of the current.

Now for the problem. Compute the average of the microscopic current density $\mathbf{j}$, and show that it can be written,

$$\langle \mathbf{j} \rangle = \mathbf{J}_f + \mathbf{J}_m + \nabla\times\mathbf{M} + \nabla\times\mathbf{N} + \frac{\partial \mathbf{P}}{\partial t} - \tfrac{1}{2}\frac{\partial \mathbf{R}}{\partial t} + \ldots, \tag{22}$$

where new definitions are the free current density,

$$\mathbf{J}_f(\mathbf{x}) = \sum_j q_{fj}\, \mathbf{v}_{fj} f(\mathbf{x} - \mathbf{x}_{fj}) = \left\langle \sum_j q_{fj}\, \mathbf{v}_{fj} \delta(\mathbf{x} - \mathbf{x}_{fj}) \right\rangle, \tag{23}$$

the molecular or ionic current density,

$$\mathbf{J}_m(\mathbf{x}) = \sum_n q_n\, \mathbf{v}_n f(\mathbf{x} - \mathbf{x}_n) = \left\langle \sum_n q_n\, \mathbf{v}_n \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle, \tag{24}$$

the magnetization,

$$\mathbf{M}(\mathbf{x}) = \sum_n \mathbf{m}_n\, f(\mathbf{x} - \mathbf{x}_n) = \left\langle \sum_n \mathbf{m}_n\, \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle, \tag{25}$$

and a "new" vector we haven't seen before,

$$\mathbf{N}(\mathbf{x}) = \sum_n \mathbf{p}_n\times\mathbf{v}_n\, f(\mathbf{x} - \mathbf{x}_n) = \left\langle \sum_n \mathbf{p}_n\times\mathbf{v}_n\, \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle. \tag{26}$$

In this procedure, please note that every time you take a spatial derivative of f, you bring in one more power of the small quantity (ratio of microscopic to mesoscopic scale). At some point you will want to neglect terms that involve second or higher derivatives of f.

Finally, define

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} - \mathbf{N} + \ldots, \tag{27}$$

so that the Maxwell equation becomes,

$$\nabla\times\mathbf{H} = \mathbf{J}_f + \mathbf{J}_m + \frac{\partial \mathbf{D}}{\partial t}. \tag{28}$$