Physics 209

Fall 2002

Homework 11

Due Friday, November 22 at 5:00pm.

This homework is due in two weeks because of the midterm. However, there may be another assignment added before the due date, on account of the material covered in lecture on Wednesday, Nov. 13. (There is only one lecture next week, since Monday is a holiday.)

About the midterm: The midterm will be in class, closed book, on Friday, November 15. It will cover material through Thomas precession. There will be a special discussion or review section on Wednesday, Nov. 13 at 5:00pm in 430 Birge. The regular discussion section on Thursday, Nov. 14 is cancelled.

1. In class we derived the covariant expression for the fields produced by an accelerated particle,

$$F^{\mu\nu} = e\left\{\frac{\Delta x^\mu b^\nu - b^\mu \Delta x^\nu}{(\Delta x \cdot u)^2} - \frac{\Delta x^\mu u^\nu - u^\mu \Delta x^\nu}{(\Delta x \cdot u)^3}\big[(\Delta x \cdot b) - 1\big]\right\}. \tag{1}$$

The notation is the following. We have chosen unit such that $c = 1$. In this problem, make life easier for yourself and set $c = 1$, until you get to final formulas where you can restore the c's by dimensional analysis. Minkowski dot products are indicated by a dot and parentheses, as in $(a \cdot b)$. The particle of charge e follows the world line $y^\mu(\tau)$. The symbol y^μ is used for the particle, while x^μ is used for a field point. For fixed x^μ, the retarded point $y^\mu(\tau_0)$ on the world line of the particle is defined by the condition,

$$\big(x - y(\tau_0)\big) \cdot \big(x - y(\tau_0)\big) = 0, \tag{2}$$

where $y^0(\tau_0) < x^0$. That is, τ_0 is the unique root of Eq. (2) satisfying $y^0(\tau_0) < x^0$. Geometrically, this says that $y^\mu(\tau_0)$ is the intersection of the world line with the backward light cone from x^μ. Equation (2) makes τ_0 a function of x^μ. The vector Δx^μ is defined by

$$\Delta x^\mu = x^\mu - y^\mu(\tau_0), \tag{3}$$

so that

$$(\Delta x \cdot \Delta x) = 0. \tag{4}$$

The vectors u^μ and b^μ are the world velocity and acceleration of the particle at the retarded point, $u^\mu = dy^\mu/d\tau$, $b^\mu = du^\mu/d\tau$, evaluated at $\tau = \tau_0$. Finally, we used the relation,

$$\frac{\partial \tau_0}{\partial x^\mu} = \frac{\Delta x^\mu}{(\Delta x \cdot u)} \tag{5}$$

in deriving Eq. (1). Equation (5) was obtained by differentiating Eq. (2) with respect to x^μ.

To convert Eq. (1) into 3 + 1 notation, use the following notation. First, write $x^\mu = (t, \mathbf{x})$, $y^\mu = (t', \mathbf{y})$, and

$$\Delta x^\mu = \begin{pmatrix} R \\ \mathbf{R} \end{pmatrix}, \tag{6}$$

where $\mathbf{R} = \mathbf{x} - \mathbf{y}(\tau_0)$, and $R = |\mathbf{R}| = t - t'$. Then write $\hat{\mathbf{n}} = \mathbf{R}/R$. Then the electric field can be written,

$$\mathbf{E}(\mathbf{x}, t) = e\Big\{\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2 R^2 (1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} + \frac{\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{cR(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3}\Big\}, \tag{7}$$

where factors of c have been restored and where $\boldsymbol{\beta} = \dot{\mathbf{y}}/c$, evaluated at the retarded time. The first major term in Eq. (7) is the "velocity field", the second, the "acceleration field." As for the magnetic field, it is given by $\mathbf{B} = \hat{\mathbf{n}} \times \mathbf{E}$.

(a) Show that the velocity field comes from the last term in Eq. (1) (the -1), and that the acceleration field comes from all the other terms. Thus, the decomposition into velocity and acceleration fields is covariant.

(b) Compute the electromagnetic stress-energy tensor $T^{\mu\nu}$ for the acceleration fields only.

(c) In the following we will need some integrals of 4-vectors over 3-dimensional surfaces in space-time. Such an integral is done in a manner that is very similar to the integration of a 3-vector over a 2-dimensional surface in ordinary space. For example, in integrating a 3-vector $\mathbf{J}$ over a 2-dimensional surface, we write

$$\int_S da\, \hat{\mathbf{n}} \cdot \mathbf{J}, \tag{8}$$

where da is an area element and $\hat{\mathbf{n}}$ is the unit vector normal to the surface. Suppose coordinates (u, v) are imposed on the surface. Then the vectors $(\partial\mathbf{x}/\partial u)\Delta u$ and $(\partial\mathbf{x}/\partial v)\Delta v$ are two small vectors (for small increments Δu and Δv) that span the plane tangent to the surface, and these define an area element

$$d\mathbf{a} = \Big(\frac{\partial\mathbf{x}}{\partial u} \times \frac{\partial\mathbf{x}}{\partial v}\Big)\Delta u \Delta v = da\, \hat{\mathbf{n}}, \tag{9}$$

which is normal to the surface. Thus the integral (8) can be written,

$$\int du\, dv\, \Big(\frac{\partial\mathbf{x}}{\partial u} \times \frac{\partial\mathbf{x}}{\partial v}\Big) \cdot \mathbf{J} = \int du\, dv\, \epsilon_{ijk} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v} J_k. \tag{10}$$

Of course, we have two choices of direction of the normal vector $\hat{\mathbf{n}}$ in Eq. (8). If the direction of $\hat{\mathbf{n}}$ in Eq. (9) is not what we want, we can swap u and v (the direction of the normal depends on the ordering of the coordinates on the surface).

Similarly, for a 3-dimensional surface in 4-dimensional space, we can impose coordinates (u, v, w) on the surface, and define the integral of K^μ over the surface by

$$\int du\, dv\, dw\, \epsilon_{\mu\nu\alpha\beta} \frac{\partial x^\mu}{\partial u} \frac{\partial x^\nu}{\partial v} \frac{\partial x^\alpha}{\partial w} K^\beta. \tag{11}$$

See Eq. (4.43) for the definition of the four-dimensional Levi-Civita symbol (in the notes on tensor analysis). When doing surface integrals in Minkowski space, it is better not to try to introduce a unit vector normal to the surface, as we did in Eq. (8), since vectors in Minkowski space may have positive, negative or zero squares (thus, a unit light-like vector makes no sense). But we can introduce a vector n^μ normal to the surface,

$$n_\beta = \epsilon_{\mu\nu\alpha\beta} \frac{\partial x^\mu}{\partial u} \frac{\partial x^\nu}{\partial v} \frac{\partial x^\alpha}{\partial w}, \tag{12}$$

whereupon the surface integral can be written

$$\int du\, dv\, dw\, n_\beta K^\beta, \tag{13}$$

we just do not attempt to normalize n^μ.

Using Eq. (11), show that if you integrate a four-vector K^μ over the 3-dimensional surface $t = 0$ in some Lorentz frame, the result is

$$\int d^3\mathbf{x}\, K^0. \tag{14}$$

This is just for practice in doing such integrals. Let the coordinates (u, v, w) on the surface be the usual (x, y, z).

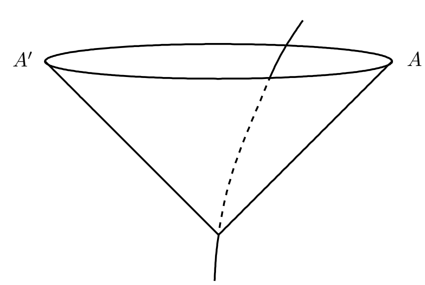

Fig. 1. Light cone produced by light emitted from one point on world line of particle.

(d) Let k^μ be a fixed vector. The idea is that by letting k^μ be a unit vector in the time-direction or spatial directions we can look at the energy or momentum radiated by the particle. Define

$$K^\mu = T^{\mu\nu} k_\nu, \tag{15}$$

and note that $\partial_\mu K^\mu = 0$.

Consider the light cone produced by the light emitted by the particle at some point on its world line, as in Fig. 1. Show that K^μ integrated over any region of the light cone vanishes, as long as the region does not contain the vertex. We exclude the vertex because it is a singular point of the cone, and the fields diverge there. To do this, show that K^μ is tangent to the light cone, so that K^μ dotted into any vector normal to the light cone is zero. Notice that the light cone is a 3-dimensional surface (it appears 2-dimensional in Fig. 1 because that is a schematic, $2+1$ diagram).

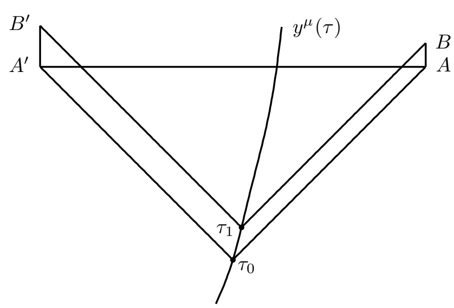

Fig. 2. Two light cones, produced by light emitted at two close proper times, τ_0 and τ_1.

(e) Now consider two closely spaced events on the world line of the particle, with proper times τ_0 and τ_1. Let $\Delta\tau = \tau_1 - \tau_0$. See Fig. 2. This figure is drawn in some Lorentz frame. Let $\Delta t'$ be the coordinate time between the two events on the world line separated by $\Delta\tau$, so that $\Delta t' = \gamma\Delta\tau$. Consider also the light cones emanating from these two events, as shown in the figure.

The points A, A' are two simultaneous events (in the selected Lorentz frame) on the first (τ_0) light cone. These two points in this $(1+1)$ figure represent the circle AA' in Fig. 1 (a $2+1$ figure), and a sphere of radius R centered on $\mathbf{y}(\tau_0)$ in a $3+1$ figure (which we can't draw). The events B, B' have the same spatial location (in the given Lorentz frame) as the events A, A', but occur at later times, namely, the time at which the light emitted at τ_1 reaches those spatial locations. Notice that B and B' are not simultaneous; the elapsed time $t_{B'} - t_{A'}$ in the figure is larger than the elapsed time $t_B - t_A$.

Find a simple expression in $3+1$ notation for $\Delta t = t_B - t_A$ as a function of $\Delta t'$ and $\hat{\mathbf{n}} = \mathbf{R}/R$ (see Eq. (6)). Note that $\hat{\mathbf{n}}$ tells where on the sphere AA' you are.

Give a physical interpretation of the integral of K^μ over the surface $ABA'B'$. You don't have to evaluate this integral, just interpret it physically. Notice that the time integration is over an infinitesimal interval, so it is trivial (but the interval depends on where on the sphere you are).

The value of this integral is a Lorentz invariant, if we regard the surface $ABA'B'$ in a geometrical sense. That is, if we choose to look at Fig. 2 in another Lorentz frame, we just evaluate components of vectors etc in the new coordinate system, but we don't change the surface (the set of events) over which we integrate. Then the integral is a Lorentz invariant, because the integrand is a Lorentz invariant.

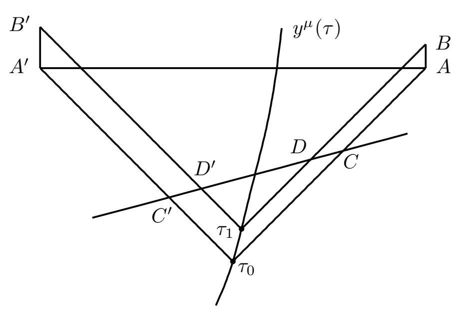

Fig. 3. A 3-dimensional hypersurface cuts through the light cones, intersecting at points CD and $C'D'$.

(f) Now consider a 3-dimensional hypersurface cutting through the two light cones, as illustrated in Fig. 3. The surface intersects the light cones at points CD and $C'D'$. The intersection $CDC'D'$ represents a 3-dimensional surface. Explain why the integral of K^μ over this surface is equal to the integral of K^μ over the surface $ABA'B'$.

(g) The new 3-dimensional hypersurface in part (f) is actually a hypersurface $t =$ const in a Lorentz frame in which the particle is at rest at τ_0. Figure 4 is Fig. 3 redrawn in this Lorentz frame, with points $CDC'D'$ marked. Evaluate the integral of K^μ over surface

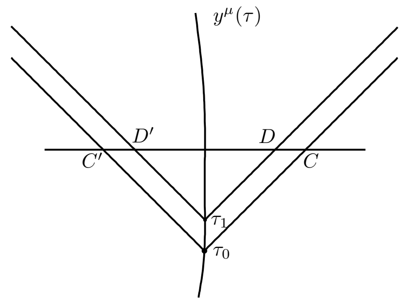

Fig. 4. Figure 3 redrawn in a Lorentz frame in which particle is at rest at τ_0.

$CDC'D'$. The answer will involve the vector k^μ, which we haven't specified yet; express your answer in terms of contractions over k^μ, and other Minkowski scalar products, as necessary, to make it obvious that the answer is a Lorentz invariant.

(h) Now use the result obtained in part (g) to find an expression for dE/dt', the energy radiated by the accelerated particle per unit retarded time, in an arbitrary Lorentz frame (the original frame in which Figs. 1–3 were drawn).

2. Jackson, problem 14.8.

3. Jackson, problem 14.9, but omit part (d) since we haven't talked about motion in inhomogeneous magnetic fields yet.

4. Jackson, problem 14.21.