Physics 209
Fall 2002
Homework 10
Due Friday, November 8 at 5:00pm.

Reading Assignment: We are now finished with Chapter 12 of Jackson. We will not cover the motion of charged particles in inhomogeneous fields. We will skip Chapter 13 (for now at least) and start on Chapter 14.

Although there are no problems in this problem set on the stress-energy tensor, you should make sure you understand how the stress-energy tensor is derived and what its properties are.

1. Jackson, Problem 12.14. Add a third part to this problem.

(c) In class it was explained that the free field Lagrangian density of the electromagnetic field must be a Lorentz scalar that is quadratic in the field. This means that $\mathcal{L}_{\rm em}$ can be either $B^2 - E^2$, $\mathbf{E}\cdot\mathbf{B}$, or $A^\mu A_\mu = \Phi^2/c^2 - \mathbf{A}^2$. We rejected $\mathbf{E}\cdot\mathbf{B}$ because it was not invariant under parity. Suppose we try it anyway, that is, suppose we take the Lagrangian density of the electromagnetic field interacting with the matter to be

$$\mathcal{L} = k\mathbf{E}\cdot\mathbf{B} - \rho\Phi + \frac{1}{c}\mathbf{J}\cdot\mathbf{A}, \tag{1}$$

where k is a constant to be determined. This Lagrangian density is written in 3+1 notation. Work out the Euler-Lagrange equations and see if they make any sense. Do this in $3+1$ notation, that is, use

$$\frac{\partial\mathcal{L}}{\partial\psi} = \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\right) + \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\right), \tag{2}$$

where ψ stands for Φ or one of the components of $\mathbf{A}$ and where $\partial_t\psi = \partial\psi/\partial t$.

2. Jackson, problem 12.19.

3. The electromagnetic field equations inherited by Maxwell were

$$\nabla\cdot\mathbf{E} = 4\pi\rho, \tag{3a}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{3b}$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{3c}$$

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \tag{3d}$$

that is, without the displacement current in Ampere's law. Maxwell realized that these equations are inconsistent with conservation of charge, although that inconsistency only shows up in time-dependent situations, since in magnetostatics $\nabla \cdot \mathbf{J} = 0$. As you probably know, Maxwell made the equations consistent by guessing the existence of the displacement current term.

Another way Maxwell could have made the equations consistent with conservation of charge would have been to replace $\mathbf{J}$ by $\mathbf{J}_\perp$, the transverse part of the current. This would not cause any disagreement with the known laws of magnetostatics, where $\nabla \cdot \mathbf{J} = 0$.

Suppose we do this. Use theorems of vector calculus to show that $\mathbf{E}$ and $\mathbf{B}$ can be expressed in terms of potentials Φ and $\mathbf{A}$, just like in standard electromagnetic theory. Then show that if the fields are produced by discrete charges, then the interaction Lagrangian for one particle moving in the fields of the other particles,

$$L_{\text{int}} = -e\Phi + \frac{e}{c}\mathbf{v} \cdot \mathbf{A}, \tag{4}$$

can be expressed completely in terms of the positions and velocities of the other particles. That is, the interactions of the particles among themselves is an action-at-a-distance theory, it can be formulated without reference to fields at all. Use Coulomb gauge to do this. Then show that the interaction Lagrangian for a system of particles interacting in this way is the interaction term of the Darwin Lagrangian.

If you need any result that is in the book or lecture notes, you may just quote it, you do not have to rederive anything. Thus, this is mostly a problem of thinking through the meaning of the theory, and there is not much calculation to do.