Physics 209
Fall 2002
Notes 1
Green's Functions

These notes contain some introductory comments on Green's functions, illustrated by the example of the Poisson equation, following the lectures in class.

First some general comments of a mathematical nature. In general, a Green's function is the response of a linear system to a point input. Green's functions can be used for both discrete and continuous systems, and a discrete example involving a network of resistors was discussed in class. In the following, however, we will consider only continuous systems.

Let $\psi(x)$ be a field of some kind, satifying a linear equation of the form,

$$(\mathcal{L}\psi)(x) = S(x), \tag{1.1}$$

where $\mathcal{L}$ is a linear operator and $S(x)$ is a "source" or "driving" term. Here x can stand for either x, $\mathbf{x}$, $(\mathbf{x}, t)$, etc., that is, we can be working in one, two, three or more dimensions, we may have a time-dependence, etc. Also, ψ and/or S may be multicomponent objects, although we will not indicate this explicitly. For example, Maxwell's equations have the form of Eq. (1.1), with ψ identified with $\mathbf{E}$ and $\mathbf{B}$, and with S identified with ρ and $\mathbf{J}$. (We are thinking of the problem of solving Maxwell's equations for given charge and current distributions.)

Equation (1.1) is an *inhomogeneous* equation, meaning that the right-hand side is non-zero. A *homogeneous* equation has a vanishing right-hand side,

$$(\mathcal{L}\phi)(x) = 0, \tag{1.2}$$

where we use a different symbol to distinguish the homogeneous solution (ϕ) from the inhomogeneous solution (ψ). Physically speaking, Eq. (1.2) means that ϕ has no sources, at least in the region of space under consideration. If the region under consideration is all of space, then usually the physics requires that $\phi = 0$, but sometimes we are thinking only of a subregion. In that case, even if Eq. (1.2) is valid in the subregion, the field ϕ may have sources outside that subregion, so a physically meaningful solution $\phi \neq 0$ may exist.

Mathematically, the solution ψ to the inhomogeneous equation (1.1) is not unique, because we can always add an arbitrary solution ϕ of the homogeneous equation (1.2) to the inhomogeneous solution, that is, $\psi \to \psi + \phi$, and clearly Eq. (1.1) will still be satisfied.

Physically, the way a unique solution is picked out of the large class of functions that satisfy Eq. (1.1) is usually by boundary conditions.

A *Green's function* is the response of a linear system to a source that concentrated at a single point. The response itself, that is, the resulting field, can be measured at any field point. Thus, the Green's function depends on two points. We shall write a Green's function generally as $G(x, x')$, where the first argument is the "field" point and the second is the "source" point. The Green's function satisfies

$$(\mathcal{L}G)(x, x') = \delta(x - x'), \tag{1.3}$$

where $\mathcal{L}$ is understood to act only on the field variable x. The source point x' is just a parameter of Eq. (1.3), which otherwise is just a special case of Eq. (1.1) (with a source term of a special form).

Given a Green's function satisfying Eq. (1.3), it is possible to write down a solution to Eq. (1.1), essentially by linear superposition. The solution is

$$\psi(x) = \int dx'\, G(x, x')S(x'). \tag{1.4}$$

It is easy to check that this satisfies Eq. (1.1).

There are a few general remarks that can be made about Green's functions that apply in most physical applications. First, since the Green's function is defined by an inhomogeneous linear equation, Eq. (1.3), it is not unique, because an arbitrary homogeneous solution can be added to it. That is, let $F(x, x')$ be a solution of

$$(\mathcal{L}F)(x, x') = 0, \tag{1.5}$$

where again x is the field variable and x' is just a parameter. Then it is clear that $G(x, x') + F(x, x')$ satisfies Eq. (1.3) if $G(x, x')$ does. This means that there is a large class of Green's functions for any given linear operator, which typically are distinguished among themselves by the boundary conditions they satisfy. Therefore one should really put subscripts or other markers on Green's function to indicate which boundary conditions they correspond to, to avoid confusion. Since this is not always done, you should always ask yourself, when you see a Green's function, what boundary conditions are intended.

Next, since the source or driving term in the defining equation (1.3) is singular at $x = x'$, the Green's function typically also has a singularity at $x = x'$. Sometimes the Green's function itself diverges or is not defined at $x = x'$, and sometimes it just fails to be smooth there, depending on the problem. On the other hand, the physics usually dictates that $G(x, x')$ be smooth at points $x \neq x'$.

Finally, we note that for reasons of convenience one often inserts a multiplicative constant on the right hand side of Eq. (1.3) when defining the Green's function (in fact, different authors use different conventions). This changes formulas like Eq. (1.4) in a trivial way.

To proceed we will specialize to the case of the Poisson equation in electrostatics. In this case we have

$$\nabla^2 \Phi = -\rho/\epsilon_0, \tag{1.6}$$

where we are working in 3-dimensional space (or some subregion of it), where ψ above is now identified with the scalar potential Φ and where S above is identified with $-\rho/\epsilon_0$. A Green's function for the Poisson equation is defined by

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}'), \tag{1.7}$$

with a conventional factor of -4π inserted on the right-hand side in comparison to Eq. (1.3). Thus, according to Eq. (1.4), a solution to the Poisson equation (1.6) is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{x}'\, G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}'). \tag{1.8}$$

Physically, the Green's function $G(\mathbf{x}, \mathbf{x}')$ is the field at $\mathbf{x}$ due to a point charge of strength $4\pi\epsilon_0$ at source point $\mathbf{x}'$. If we are working in all of 3-dimensional space, then we know what this potential must be; it is given by Coulomb's law,

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \tag{1.9}$$

This is the *free-space* Green's function for the Poisson equation, and mathematically speaking it is the unique solution to Eq. (1.7) that vanishes as $|\mathbf{x}| \to \infty$. Notice that it is singular at $\mathbf{x} = \mathbf{x}'$. This is a specific example (the simplest example) of boundary conditions for the Poisson equation. When we use the free-space Green's function in Eq. (1.8), we obtain

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{x}'\, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \tag{1.10}$$

which is just Coulomb's law again, applied to a charge distribution. The total potential is a linear superposition of point potentials, integrated over the source.

Thus, in free space, the Green's function formalism does not give us anything terribly new, although if we wished to derive Coulomb's law from Maxwell's equations (the inverse of the historical order), we could say that Coulomb's law comes from the Green's function for the electrostatic problem.

Let us now consider the Poisson equation (1.6) restricted to some region or volume V of space, with a boundary or surface S. We let $\hat{\mathbf{n}}$ be the unit normal at the surface S of

V, pointing outwards (away from V). The example of the multi-wire proportional chamber was mentioned in class; in this example, the volume V is finite and the boundaries are conductors. In general, the volume V may be finite or infinite (in the latter case it may be convenient to regard V as the limit of a closed region that expands to infinite volume), and it may be topologically non-trivial (for example, with excluded regions in the middle, etc). The boundary of V need not be made of conductors or pieces of conductors, although if it is (as often happens in practice), then the potential Φ is piecewise constant on the boundary.

Let us assume that the charge density ρ is given in the interior of the region V. Then the Poisson equation possesses a *uniqueness theorem*, which states that if Φ is given on the boundary of V, then there exists a unique solution in the interior satisfying the given boundary conditions. These are called *Dirichlet* boundary conditions (Φ given on the boundary). The multi-wire proportional chamber example was an example of Dirichlet boundary conditions. Another version of the uniqueness theorem states that there is a unique solution (to within an additive constant) if the normal derivative $\partial\Phi/\partial n$ is given on the boundary. These are called *Neumann* boundary conditions (the normal derivative of Φ, or the normal component of the electric field, is given on the boundary). In fact, the solution is unique if Φ is given on parts of the boundary and $\partial\Phi/\partial n$ is given on other parts. These are called *mixed* boundary conditions. Dirichlet boundary conditions are the most common in practice, and we will probably concentrate on these. The uniqueness theorem (in its various versions) is not hard to prove, it is in Jackson and undergraduate books.

Let us concentrate now on Dirichlet boundary conditions for the Poisson equation. Thus, we will assume that V and its surface S with outward normal $\hat{\mathbf{n}}$ are given, that $\rho(\mathbf{x})$ is known inside V, that Φ is given on the boundary, and that we wish to find Φ in the interior. In this case it turns out to be convenient to define the *Dirichlet Green's function* $G_D(\mathbf{x}, \mathbf{x}')$, defined by Eq. (1.7) (all Green's functions satisfy this equation), and the boundary conditions

$$G(\mathbf{x}, \mathbf{x}') = 0 \qquad \text{when} \qquad \mathbf{x} \in S. \tag{1.11}$$

That is, G_D vanishes when the field point is on the boundary.

The boundary conditions on $G_D(\mathbf{x}, \mathbf{x}')$ could be achieved physically by replacing the boundary, if necessary, by grounded conductors, although this is only a matter of making a physical interpretation of the function $G_D(\mathbf{x}, \mathbf{x}')$. The actual, physical boundary surfaces we have in mind when solving the Poisson equation for Φ do not (as mentioned previously) have to be conductors. But this gives us a physical interpretation of the function $G_D(\mathbf{x}, \mathbf{x}')$: it is the potential at field point $\mathbf{x}$ due to a point source of strength $4\pi\epsilon_0$ at location $\mathbf{x}'$,

in the presence of grounded conductors for boundaries. The presence of the grounded conductors modifies the potential produced by the point charge, and in particular the field lines emanating from $\mathbf{x}'$ that terminate on the conductors do so at right angles to the surface (since the surface is an equipotential $\Phi = 0$ and the electric field is purely normal there). In fact, we know that if we bring up a point charge into the neighborhood of grounded conductors, there will be charges that come in from infinity (along the grounding wires) and spread themselves over the surface of the conductors, in order to make the total electric field, that produced by the point charge we brought in and that produced by the surface charges, be orthogonal to the surface at the surface. Think of a point charge in the neighborhood of an infinite, grounded plane (the simplest image charge problem).

Because of the grounded conductors, the function $G_D(\mathbf{x}, \mathbf{x}')$ is not the same as the free-space Green's function (1.9). However, the difference between the two must be a solution of the homogeneous equation (the Laplace equation), that is, we must have

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}'), \tag{1.12}$$

where

$$\nabla^2 F(\mathbf{x}, \mathbf{x}') = 0. \tag{1.13}$$

Notice that ∇^2 in this equation only acts on the field point $\mathbf{x}$, while the point $\mathbf{x}'$ is just a parameter. The physical meaning of $F(\mathbf{x}, \mathbf{x}')$ is now clear; since it satisfies the Laplace equation in the volume V, it must be the potential produced by charges outside V. In fact, it is the field produced by the surface charges on the grounded conductors, in the physical interpretation of $G_D(\mathbf{x}, \mathbf{x}')$ provided in the previous paragraph. Since the distribution of these charges depends on the position $\mathbf{x}'$ of the point charge (the source of G_D), the function F must depend on $\mathbf{x}'$, too.

A parenthetical remark here is that sometimes we are given some potential that satisfies the Laplace equation in some region of space, and we would like to know what sources outside that region produce the potential. For example, we may sample the gravitational field of the earth out in space (for example, by the orbits of satellites), and we may wish to obtain information about the mass distribution within the earth that produced this field. Then it turns out that the sources that produce the given field are not unique. This can be seen in simple examples such as spherically symmetric distributions of charge (outside the charge radius only the total charge can be determined), or in the case of a point charge in the presence of a conducting plane (the image charge and the surface charge produce the same field in the region above the plane). So in the above analysis, we shouldn't attach too much importance to the sources of $F(\mathbf{x}, \mathbf{x}')$, but the model of the grounded conductors in the

presence of the point source at least gives us one example of charges (the surface charges) that could produce F.

In any case, we now wish to show how the Dirichlet Green's function $G_D(\mathbf{x}, \mathbf{x}')$ can be used to solve the Poisson equation for Φ. For this we call on Green's second theorem, proven in class and also in the book. This theorem states that if $\psi(\mathbf{x})$ and $\phi(\mathbf{x})$ are any two fields, then

$$\int_V d\mathbf{x} \left(\psi \nabla^2 \phi - \phi \nabla^2 \psi\right) = \int_S da \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right), \tag{1.14}$$

where V is a volume and S is its surface.

In class we first used this theorem to show that the Dirichlet Green's function is symmetrical in the source and field points,

$$G_D(\mathbf{x}, \mathbf{y}) = G_D(\mathbf{y}, \mathbf{x}), \tag{1.15}$$

where $\mathbf{x}$ and $\mathbf{y}$ are any two points in V. This symmetry is a special property of the Dirichlet Green's function of the Poisson equation, and does not apply in other contexts (for example, the Green's function for the Schrödinger equation is not in general symmetric). But it means that for Dirichlet problems for the Poisson equation, you can ignore the distinction between field and source points. Nevertheless, I think it is better to keep these two distinct, not only because of the other contexts, but more importantly because of the physical interpretation of the Green's function as the field at the field point produced by a point source at the source point. In fact, the property (1.15) is subtle; it is not obvious, and I do not know of a simple physical interpretation of it.

By the way, Jackson defines the Green's function differently than I have done. His definition is

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}'), \tag{1.16}$$

that is, with the operator acting on the source point. In view of the symmetry relation (1.15), the two definitions are equivalent, but if you were talking about Green's functions in other contexts, you would want to let the operator act on the field point, not the source point. Jackson does this because he only wants to talk about the Green's function for the Poisson equation, and there it does not matter. Also, there is the question of the symmetry of the Neumann Green's function, which we have not discussed here.

In class we used Green's second theorem a second time to obtain the general solution of the Dirichlet problem for Φ,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V d\mathbf{x}'\, G_D(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') - \frac{1}{4\pi} \int_S da'\, \frac{\partial}{\partial n'} G_D(\mathbf{x}, \mathbf{x}')\Phi(\mathbf{x}'), \tag{1.17}$$

which is Jackson Eq. (1.44). The same derivation is in the book. This equation gives Φ in the interior in terms of ρ in the interior and the given values of Φ on the boundary (in the second integral).

Equation (1.17) gives the potential Φ in terms of the Green's function G_D, and we must ask whether it is any easier to determine G_D than it is to determine Φ directly. The answer is that it may be hard to determine G_D; the methods for finding it, such as separation of variables, image charges etc., are the same ones we would use to find Φ directly. However, G_D satisfies simpler boundary conditions than does Φ (G_D vanishes on the boundary, whereas the specified values of Φ on the boundary need not be zero). In fact, we can see that G_D depends only on the region V, but does not depend in the charge distribution ρ in the interior or the values of Φ on the boundary we wish our potential to satisfy. Thus, if G_D is known for one region of space, it can be used for all Dirichlet problems in that region (with different ρ in the interior and different given Φ on the boundary). Using the Green's function as in Eq. (1.17) implies a neat division of the labor. Moreover, there are cases in which the Green's function can be determined rather easily (such as by the method of images), and Eq. (1.17) then leads to nontrivial results.

Let us now interpret Eq. (1.17) physically. The first term looks like the Coulomb integral (1.10), and it has the interpretation of being the potential produced by charges $\rho(\mathbf{x}')$ brought in from infinity in the presence of grounded conductors for the surface of V. In this case, the total potential is forced to vanish on the boundary (like G_D), and the solution is just a linear superposition of the point solutions contained in G_D.

The second term in Eq. (1.17) is necessary to make the potential Φ take on arbitrary, specified (and nonzero) values on the boundary. It turns out to have an interpretation as the field produced by a dipole layer adjacent to the boundary.

A *dipole layer* can be thought of as two nearby charge layers with charge densities $\sigma(\mathbf{x})$ and $-\sigma(\mathbf{x})$, separated by distance ϵ, taken in the limit that $\epsilon \to 0$, $\sigma \to \infty$, in such a way that the product $\tau = \sigma\epsilon$ (at a fixed $\mathbf{x}$ point) remains fixed. Then τ is interpreted as the dipole moment per unit area. The dipole density τ can be a function of $\mathbf{x}$ along the surface.

As you know, the normal component of the electric field is discontinuous across a charge layer. If the $\hat{\mathbf{n}}$ is a normal to the charge layer, pointing from side 2 to side 1, then the discontinuity in the normal component of the electric field is given by

$$E_{n1} - E_{n2} = \sigma/\epsilon_0. \tag{1.18}$$

Similarly, there is a discontinuity in the potential across a dipole layer. If the normal to a dipole layer is the unit vector $\hat{\mathbf{n}}$ passing from the negative to the positive side, then the

discontinuity is

$$\Phi_1 - \Phi_2 = \tau/\epsilon_0, \tag{1.19}$$

where again, $\hat{\mathbf{n}}$ points from side 2 to side 1. The electric field is continuous across a dipole layer (it jumps up on crossing the first charge layer, but then jumps back again on crossing the second one).

The potential at field point $\mathbf{x}$ produced by a dipole layer along surface S_d is given by the integral,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_{S_d} da'\sigma(\mathbf{x}')[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}' - \epsilon\hat{\mathbf{n}}')], \tag{1.20}$$

where $\hat{\mathbf{n}}'$ is the normal to the dipole surface S_d at position $\mathbf{x}'$, going from the negative to the positive side, and where G is the Green's function for the problem at hand. For example, if the dipole layer is in free space (no conductors nearby), then G is the free-space Green's function. If there are grounded conductors nearby, then G is the Dirichlet Green's function G_D (to take care of the induced charges that appear on the conductors when the dipole layer is brought up). In the limit $\epsilon \to 0$, $\sigma \to \infty$, we have

$$G(\mathbf{x},\mathbf{x}' - \epsilon\hat{\mathbf{n}}') = -\epsilon\hat{\mathbf{n}}'\cdot\nabla' G(\mathbf{x},\mathbf{x}'), \tag{1.21}$$

and $\epsilon\sigma(\mathbf{x}') = \tau(\mathbf{x}')$, so the integral (1.20) becomes

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_{S_d} da'\,\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')\tau(\mathbf{x}'). \tag{1.22}$$

Now suppose we have a Dirichlet problem for Φ, in which Φ is required to vanish on the boundary, and suppose the charges inside the volume V consist of some continuous distribution $\rho(\mathbf{x}')$ plus a dipole layer on a surface S_d somewhere in the interior. Then the potential $\Phi(\mathbf{x})$ inside V is given by two terms, one being the first integral of Eq. (1.17) and the other being the integral of Eq. (1.22), with G identified with G_D. If now the dipole layer is pushed very close to the boundary of V, we will still have $\Phi = 0$ exactly on the boundary, but a short distance away we cross the dipole layere and the potential jumps discontinuously to the value $-\tau/\epsilon_0$. Then by choosing the dipole density τ at the boundary point $\mathbf{x}'$ to be $-\Phi(\mathbf{x}')\epsilon_0$, where $\Phi(\mathbf{x}')$ is some (generally nonzero) desired value of Φ at the boundary, we make this desired value of Φ appear just across the dipole layer. In the limit the dipole layer is pushed up against the boundary, we have a solution of the Poisson equation in the interior which approaches the desired (generally nonzero) values of Φ on the boundary. Moreover, the substitution of $\tau(\mathbf{x}') = -\Phi(\mathbf{x}')\epsilon_0$ in Eq. (1.21) produces the second integral of Eq. (1.17). In this way we obtain a physical interpretation (or at least a physical model) for both terms in Eq. (1.17).