Physics 221A
Fall 1996
Notes 14
Coupling of Angular Momenta

In these notes we will discuss the problem of the coupling or addition of angular momenta. It is assumed that you have all had experience with this in undergraduate school, and that, for example, you know how to compute simple Clebsch-Gordan coefficients. Therefore these notes will concentrate on the fundamental principles, and on subjects which you probably have not seen before.

We begin with the tensor product, which is a mathematical operation used in quantum mechanics to combine together the ket spaces corresponding to different degrees of freedom to obtain a ket space for a composite system. For example, one can combine the ket spaces for two individual particles, to obtain the ket space for a two-particle system; or one can combine the orbital and spin degrees of freedom for a single particle.

To be specific, suppose we have two spinless distinguishable particles, labeled 1 and 2, and let the ket spaces for these particles be denoted $\mathcal{E}_1$ and $\mathcal{E}_2$. These ket spaces can be identified with spaces of wave functions on 3-dimensional space, so we write

$$\begin{aligned}\mathcal{E}_1 &= \{\phi(\mathbf{r}),\ \text{particle 1}\}\\ \mathcal{E}_2 &= \{\chi(\mathbf{r}),\ \text{particle 2}\}.\end{aligned} \tag{14.1}$$

We regard these two ket spaces as two distinct spaces, because they are associated with two different particles. The use of two symbols (ϕ and χ) for the wave functions is just a way of reminding ourselves which particle is being referred to. Now the wave function space for the combined, two-particle system is another space, the space $\mathcal{E}$ of wave functions defined on the combined, 6-dimensional configuration space $(\mathbf{r}_1, \mathbf{r}_2)$:

$$\mathcal{E} = \{\psi(\mathbf{r}_1, \mathbf{r}_2)\}. \tag{14.2}$$

A special case of a two-particle wave function is a product of single particle wave functions,

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \phi(\mathbf{r}_1)\chi(\mathbf{r}_2), \tag{14.3}$$

but not every two-particle wave function can be written in this form. On the other hand, every two-particle wave function can be written as a linear combination of products of single particle wave functions. To see this, we simply introduce a basis $\{\phi_n(\mathbf{r})\}$ of wave functions in $\mathcal{E}_1$, and another basis $\{\chi_m(\mathbf{r})\}$ of wave functions in $\mathcal{E}_2$. Then an arbitrary two-particle wave function can be written,

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n,m} c_{nm}\, \phi_n(\mathbf{r}_1)\, \chi_m(\mathbf{r}_2), \tag{14.4}$$

where the c_{nm} are expansion coefficients. In other words, the products of single particle basis wave functions forms a basis in the wave function space for two particles.

In the construction we have just presented, we say that the space $\mathcal{E}$ is the *tensor product* of the spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, and we write

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2. \tag{14.5}$$

Loosely speaking, one can say that the tensor product space is the space spanned by the products of wave functions from the two constituent spaces. In addition to forming the tensor product of ket spaces, one can also form the tensor product of kets. An example is given in wave function language by Eq. (14.3), which in ket language would be written

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle. \tag{14.6}$$

Thus, the tensor product of kets corresponds to the ordinary product of wave functions. Often in casual physics notation, the tensor product sign $\otimes$ is omitted from a tensor product such as (14.6), and one simply writes $|\phi\rangle|\chi\rangle$.

More generally, suppose $\mathcal{E}_1$ is a ket space spanned by the basis $\{|\alpha_n\rangle\}$, and $\mathcal{E}_2$ is a ket space spanned by the basis $\{|\beta_m\rangle\}$. Then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a new ket space spanned by the basis kets $\{|\alpha_n\rangle \otimes |\beta_m\rangle\}$. Obviously, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are finite-dimensional, then so is $\mathcal{E}$, and we have

$$\dim \mathcal{E} = (\dim \mathcal{E}_1)(\dim \mathcal{E}_2). \tag{14.7}$$

If either $\mathcal{E}_1$ or $\mathcal{E}_2$ is infinite-dimensional, then so is $\mathcal{E}$.

An important example of the tensor product occurs when we combine the spatial or orbital degrees of freedom of a single particle with the spin degrees of freedom. We can define orbital and spin ket spaces by

$$\begin{aligned} \mathcal{E}_{\rm orb} &= \mathrm{span}\{|\mathbf{r}\rangle\}, \\ \mathcal{E}_{\rm spin} &= \mathrm{span}\{|m\rangle\}, \end{aligned} \tag{14.8}$$

where $m = -s, \ldots, +s$. Notice that $\mathcal{E}_{\rm orb}$ is infinite-dimensional, whereas $\mathcal{E}_{\rm spin}$ is finite-dimensional. Then the total Hilbert space for the particle is

$$\mathcal{E} = \mathcal{E}_{\rm orb} \otimes \mathcal{E}_{\rm spin} = \mathrm{span}\{|\mathbf{r}\rangle \otimes |m\rangle\}. \tag{14.9}$$

Let us write

$$|\mathbf{r}, m\rangle = |\mathbf{r}\rangle \otimes |m\rangle \tag{14.10}$$

for the basis vectors of the tensor product space, so that an arbitrary ket $|\psi\rangle$ belonging to $\mathcal{E}$ can be written as a linear combination of these basis vectors. Then we have

$$|\psi\rangle = \sum_m \int d^3\mathbf{r}\, |\mathbf{r}, m\rangle\langle\mathbf{r}, m|\psi\rangle = \sum_m \int d^3\mathbf{r}\, |\mathbf{r}, m\rangle \psi_m(\mathbf{r}), \tag{14.11}$$

where

$$\psi_m(\mathbf{r}) = \langle\mathbf{r}, m|\psi\rangle. \tag{14.12}$$

These operations illustrate how we can go back and forth between ket language and wave function language for a particle with spin; the wave function $\psi_m(\mathbf{r})$ is a multi-component or spinor wave function, which one can imagine as a column vector. For example, in the case of a spin-$\frac{1}{2}$ particle, we can write

$$\psi_\pm(\mathbf{r}) = \begin{pmatrix} \psi_+(\mathbf{r}) \\ \psi_-(\mathbf{r}) \end{pmatrix}. \tag{14.13}$$

One can also form the tensor product of operators. Suppose A_1 is an operator which acts on $\mathcal{E}_1$, and A_2 is an operator which acts on $\mathcal{E}_2$, and let $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$. Then we define an operator $A_1 \otimes A_2$, whose action on a tensor product of vectors from $\mathcal{E}_1$ and $\mathcal{E}_2$ is given by

$$(A_1 \otimes A_2)(|\alpha\rangle_1 \otimes |\beta\rangle_2) = (A_1|\alpha\rangle_1) \otimes (A_2|\beta\rangle_2). \tag{14.14}$$

In this equation, the subscripts 1, 2 on the kets indicate which space ($\mathcal{E}_1$ or $\mathcal{E}_2$) the kets belong to. But since an arbitrary vector in $\mathcal{E}$ can be represented as a linear combination of tensor products of vectors from $\mathcal{E}_1$ and $\mathcal{E}_2$, we can use linear superposition to extend Eq. (14.14) to define the action of $A_1 \otimes A_2$ on an arbitrary vector in $\mathcal{E}$. Again, in casual physics notation, the tensor product sign is often omitted in a product of operators such as $A_1 \otimes A_2$, and one would simply write $A_1 A_2$.

A special case of the above is when one or the other of the operators A_1 or A_2 is the identity. For example, if $A_2 = 1$, we have

$$(A_1 \otimes 1)(|\alpha\rangle_1 \otimes |\beta\rangle_2) = (A_1|\alpha\rangle_1) \otimes |\beta\rangle_2. \tag{14.15}$$

In such cases it would be normal in casual physics notation to write simply A_1 instead of $A_1 \otimes 1$, thereby confusing an operator which acts on $\mathcal{E}_1$ with an operator which acts on $\mathcal{E}$. Similar considerations apply to operators of the form $1 \otimes A_2$, where $A_1 = 1$. We note that operators of the type A_1, A_2, when regarded as acting on the tensor product space $\mathcal{E}$, always commute with one another,

$$[A_1 \otimes 1, 1 \otimes A_2] = [A_1, A_2] = 0, \tag{14.16}$$

as follows from Eq. (14.14). As one says, A_1 and A_2 commute because they act on different spaces.

Now we proceed to the problem of the addition of angular momenta. Suppose we have two ket spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ upon which two angular momentum operators $\mathbf{J}_1$ and $\mathbf{J}_2$ act, each of which satisfies the angular momentum commutation relations (11.1). The case of the orbital and spin ket spaces discussed above is a good example; if $\mathcal{E}_1 = \mathcal{E}_{\rm orb}$ and $\mathcal{E}_2 = \mathcal{E}_{\rm spin}$, then the angular momentum $\mathbf{J}_1$ is the orbital angular momentum $\mathbf{L}$ and $\mathbf{J}_2$ is the spin angular momentum $\mathbf{S}$. Then in accordance with the general theory laid out in Notes 11, we know that each space $\mathcal{E}_1$ and $\mathcal{E}_2$ breaks up into the direct sum of a sequence of irreducible subspaces, each with a definite j value. For example, an irreducible subspace of $\mathcal{E}_{\rm orb}$ is spanned by wave functions of the form $u_n(r)Y_{\ell m}(\theta,\phi)$ for a definite radial wave function $u_n(r)$, a definite value of ℓ, and for $-\ell \le m \le +\ell$. This subspace has dimensionality $2\ell+1$. As for the space $\mathcal{E}_{\rm spin}$, it consists of a single irreducible subspace of dimensionality $2s+1$, characterized by the value s of the spin.

Because both spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ can be decomposed into irreducible subspaces, we can study these irreducible subspaces one at a time when taking the tensor product of $\mathcal{E}_1$ and $\mathcal{E}_2$. In other words, without loss of generality, we can assume that both $\mathcal{E}_1$ and $\mathcal{E}_2$ consist of a single irreducible subspace. If this is not so at the start, we can redefine $\mathcal{E}_1$ and $\mathcal{E}_2$ to be irreducible subspaces of their former selves, if necessary. For example, in the case of the orbital ket space, we may wish to let $\mathcal{E}_1$ represent the 5-dimensional space spanned by the $Y_{\ell m}$'s (with definite radial wave function) for $\ell = 2$. With these assumptions, spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ are characterized by definite angular momentum values j_1 and j_2, and their dimensionalities are $2j_1+1$ and $2j_2+1$, respectively.

We now consider the tensor product space $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$, which has dimensionality $(2j_1+1)(2j_2+1)$. We define a "total" angular momentum operator acting on $\mathcal{E}$ by

$$\mathbf{J} = \mathbf{J}_1 \otimes 1 + 1 \otimes \mathbf{J}_2 = \mathbf{J}_1 + \mathbf{J}_2, \tag{14.17}$$

where the final expression is in casual physics notation. We note that $\mathbf{J}_1$ and $\mathbf{J}_2$ commute,

$$[\mathbf{J}_1, \mathbf{J}_2] = 0, \tag{14.18}$$

since they act on different spaces. (This commutator means that every component of $\mathbf{J}_1$ commutes with every component of $\mathbf{J}_2$.) From this it easily follows that $\mathbf{J}$ also satisfies the angular momentum commutation relations,

$$[J_i, J_j] = i\hbar\, \epsilon_{ijk}\, J_k. \tag{14.19}$$

Therefore we have a new space $\mathcal{E}$ upon which a vector $\mathbf{J}$ of angular momentum operators acts; in accordance with the theory of Notes 11, this space breaks up into the direct sum of a sequence of irreducible subspaces, each characterized by some j value. Furthermore, we know that there exists a standard basis $|\alpha jm\rangle$ on $\mathcal{E}$. The basic problem of the addition of angular momenta is to find which values of j occur in $\mathcal{E}$ and with what multiplicity, and to find some convenient way of constructing the standard basis $|\alpha jm\rangle$.

Since the spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ are acted upon by angular momentum operators $\mathbf{J}_1$ and $\mathbf{J}_2$, these spaces possess standard bases, say, $\{|j_1 m_1\rangle\}$ and $\{|j_2 m_2\rangle\}$, where j_1 and j_2 are fixed numbers characterizing the spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ and where there is no need for an α index since by assumption $\mathcal{E}_1$ and $\mathcal{E}_2$ were irreducible. [For example, the kets $|j_1 m_1\rangle$ are simultaneous eigenkets of J_1^2 and J_{1z} with eigenvalues $j_1(j_1+1)\hbar^2$ and $m_1\hbar$, respectively.] Therefore we can form a basis in $\mathcal{E}$ by taking the tensor products of basis vectors in $\mathcal{E}_1$ and $\mathcal{E}_2$. We introduce a shorthand notation for these tensor product basis kets in $\mathcal{E}$, writing

$$|j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle, \tag{14.20}$$

where of course $m_1 = -j_1, \ldots, j_1$ and $m_2 = -j_2, \ldots, j_2$. We will call the basis $\{|j_1 j_2 m_1 m_2\rangle\}$ in $\mathcal{E}$ the *tensor product basis* or *uncoupled basis.* The tensor product basis is not the same as the standard basis in $\mathcal{E}$.

We will denote the vectors of the standard basis in $\mathcal{E}$ by $|\alpha jm\rangle$, as in the general theory. As it turns out, however, the index α is unnecessary, and we will eventually be able to write simply $|jm\rangle$. We will also refer to this basis as the *coupled basis.* The vectors of the standard or coupled basis are simultaneous eigenkets of J^2 and J_z. To find these kets, we begin by looking for eigenkets of J_z. This is easy, because the vectors of the tensor product basis are all eigenkets of $J_z = J_{1z} + J_{2z}$, with eigenvalues $(m_1+m_2)\hbar$:

$$\begin{aligned} J_z|j_1 j_2 m_1 m_2\rangle &= (J_{1z}+J_{2z})|j_1 m_1\rangle|j_2 m_2\rangle = (m_1+m_2)\hbar|j_1 j_2 m_1 m_2\rangle \\ &= m\hbar|j_1 j_2 m_1 m_2\rangle, \end{aligned} \tag{14.21}$$

where we set

$$m = m_1 + m_2, \tag{14.22}$$

for the quantum number of J_z. The spectrum of J_z ranges from the maximum value of m_1+m_2, which is j_1+j_2, down to the minimum, which is $-(j_1+j_2)$.

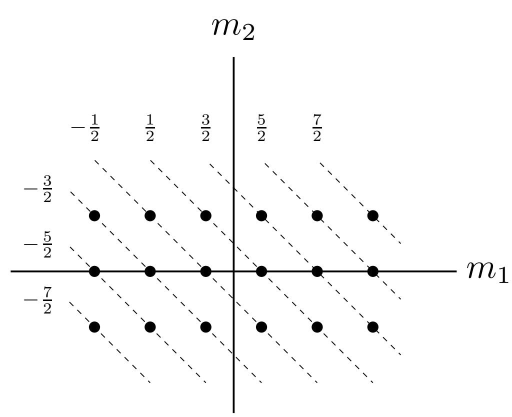

Fig. 14.1. Each dot in the rectangular array stands for one vector of the uncoupled or tensor product basis, $|j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle$. The dashed lines are contours of $m = m_1 + m_2$.

These eigenvalues of J_z are in general degenerate. To follow the subsequent argument, it helps to have an example. Let us take the case $j_1 = \frac{5}{2}$ and $j_2 = 1$, so that $2j_1 + 1 = \dim \mathcal{E}_1 = 6$ and $2j_2 + 1 = \dim \mathcal{E}_2 = 3$. Thus, the dimensionality of $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is $6 \times 3 = 18$. It is convenient to make a plot in the m_1-m_2 plane of the basis vectors of the tensor product basis, by placing a dot at each allowed m_1 and m_2 value. We then obtain a rectangular array of dots, as illustrated in Fig. 1. As illustrated in the figure, lines of constant $m = m_1 + m_2$ are straight lines, dashed in the figure, sloping downwards. The number of dots each dashed line passes through is the number of kets of the uncoupled basis with a given m value; we see that the degeneracies of the different m values range from one to three, as summarized in Table 14.1.

In particular, the stretched state [the one for which $(m_1, m_2) = (j_1, j_2)$, in the upper right hand corner in the figure] is a nondegenerate eigenstate of J_z with quantum number $m = \frac{7}{2}$. But since $[J^2, J_z] = 0$, this state is also an eigenstate of J^2, in accordance with Theorem 1.2. But what is the eigenvalue of J^2, i.e., what is the quantum number j? Certainly we cannot have $j < \frac{7}{2}$, because then this would violate the rule $m \le j$. Nor can we have $j > \frac{7}{2}$, because, for example, if the stretched state were the state $|jm\rangle = |\frac{9}{2}\frac{7}{2}\rangle$, then we could apply the raising operator J_+ and obtain the state $|\frac{9}{2}\frac{9}{2}\rangle$. But there is no state with $m = \frac{9}{2}$, as we see from the figure or table. Therefore the j quantum number of the stretched state must be $j = \frac{7}{2}$; this state is the state $|jm\rangle = |\frac{7}{2}\frac{7}{2}\rangle$ of the coupled basis. But given this state, we can apply lowering operators J_- to obtain all eight states $|\frac{7}{2}m\rangle$, which are indicated in the third column of the table.

m	$g(m)$	$j=\frac{7}{2}$	$j=\frac{5}{2}$	$j=\frac{3}{2}$
$\frac{7}{2}$	1	1		
$\frac{5}{2}$	2	1	1	
$\frac{3}{2}$	3	1	1	1
$\frac{1}{2}$	3	1	1	1
$-\frac{1}{2}$	3	1	1	1
$-\frac{3}{2}$	3	1	1	1
$-\frac{5}{2}$	2	1	1	
$-\frac{7}{2}$	1	1		
Total	18	8	6	4

Table 14.1. The first column contains m, the quantum number of J_z; the second column contains $g(m)$, the degeneracy of m; columns 3, 4 and 5 contain a unit for each vector $|jm\rangle$ of the standard or coupled basis with given j and m values.

Now let us consider the 2-dimensional eigenspace of J_z corresponding to quantum number $m = \frac{5}{2}$. This space is spanned by the kets of the uncoupled basis corresponding to $(m_1, m_2) = (\frac{5}{2}, 0)$ and $(\frac{3}{2}, 1)$. Furthermore, the ket $|\frac{7}{2}\frac{5}{2}\rangle$ of the coupled basis also lies in this space. Let us consider the ket, call it $|x\rangle$, which is orthogonal to $|\frac{7}{2}\frac{5}{2}\rangle$ in this space. Certainly $|x\rangle$ is an eigenket of J_z with eigenvalue $\frac{5}{2}$. And it is also an eigenket of J^2, as a simple extension of the proof of Theorem 1.2 will show. But what is the j value? Certainly we cannot have $j < \frac{5}{2}$, because this would violate the $m \le j$ rule. Nor can we have $j > \frac{5}{2}$, because if we had $j = \frac{7}{2}$, for example, for the state $|x\rangle$, then we would have two linearly independent states, both with $m = \frac{5}{2}$ and $j = \frac{7}{2}$. We could then apply the raising operator J_+ to both of them, and obtain two linearly independent states with $m = \frac{7}{2}$, $j = \frac{7}{2}$. But there is only one state with $m = \frac{7}{2}$, as we see from the figure or table. Therefore we must have $j = \frac{5}{2}$ for the state $|x\rangle$, which otherwise is the ket $|\frac{5}{2}\frac{5}{2}\rangle$ of the coupled basis. Then, by applying lowering operators to this, we obtain all six vectors $|\frac{5}{2}m\rangle$, which are indicated in the fourth column of the table. Finally, we carry out the same procedure for the 3-dimensional space corresponding to $m = \frac{3}{2}$, and we obtain four more vectors $|\frac{3}{2}m\rangle$ of the coupled basis.

In this way, all 18 dimensions of $\mathcal{E}$ are used up, as indicated by the totals at the bottom of the table. We see that the tensor product space $\mathcal{E}$ consists of the direct sum of three irreducible subspaces, corresponding to $j = \frac{3}{2}$, $\frac{5}{2}$, and $\frac{7}{2}$, and that each of these j values occurs with multiplicity one. These facts are summarized by the notation,

$$\tfrac{5}{2} \otimes 1 = \tfrac{3}{2} \oplus \tfrac{5}{2} \oplus \tfrac{7}{2}, \tag{14.23}$$

which corresponds to the dimensionality count,

$$18 = 6 \times 3 = 4 + 6 + 8. \tag{14.24}$$

By using diagrams like these, it is easy to work out the general case in which we combine arbitrary angular momenta j_1 and j_2. The result is

$$j_1 \otimes j_2 = |j_1 - j_2| \oplus |j_1 - j_2| + 1 \oplus \ldots \oplus j_1 + j_2, \tag{14.25}$$

that is, the j values in $j_1 \otimes j_2$ range from a minimum of $|j_1 - j_2|$ to a maximum of $j_1 + j_2$ in integer steps, and each j value in this range occurs once. Since no j value occurs more than once, there is no need for the index α, and the vectors of the coupled basis can be denoted simply $|jm\rangle$. Also, since the dimensionalities of the subspaces must add up to the dimensionality of the tensor product space, we have the identity

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} 2j + 1 = (2j_1 + 1)(2j_2 + 1). \tag{14.26}$$

This identity can be proved by elementary algebra, as a check on the count of dimensions.

At this point we have two bases in $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$, the uncoupled basis $|j_1 j_2 m_1 m_2\rangle$ with $-j_1 \le m_1 \le j_1$ and $j_2 \le m_2 \le j_2$ and the coupled basis $|jm\rangle$ with $|j_1 - j_2| \le j \le j_1 + j_2$ and $-j \le m \le j$. These two bases must be connected by a unitary matrix, the components of which are just the scalar products of the vectors from one basis with the vectors from the other. That is, we have

$$|jm\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | jm\rangle, \tag{14.27a}$$

$$|j_1 j_2 m_1 m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |jm\rangle \langle jm | j_1 j_2 m_1 m_2\rangle, \tag{14.27b}$$

which can be regarded as the insertion of two resolutions of the identity in two different ways. The expansion coefficients $\langle j_1 j_2 m_1 m_2 | jm\rangle$ or $\langle jm | j_1 j_2 m_1 m_2\rangle$ are called the *Clebsch-Gordan* coefficients, or, more properly, the *vector coupling* coefficients.

The Clebsch-Gordan coefficients can be calculated in a straightforward way by using lowering operators and by constructing states (such as $|x\rangle$ in the discussion above) which are orthogonal to all known states in a subspace of given m. In this process, it is necessary to make certain phase conventions; the standard is to follow the phase conventions discussed in Notes 11, in which the matrix elements of $J_\pm$ in the standard basis are real and positive, and to require in addition

$$\langle jj|j_1j_2j_1, j-j_1\rangle > 0, \tag{14.28}$$

for each allowed j value. Under these phase conventions, the Clebsch-Gordan coefficients are real, i.e.,

$$\langle jm|j_1j_2m_1m_2\rangle = \langle j_1j_2m_1m_2|jm\rangle, \tag{14.29}$$

so that the unitary matrix connecting the coupled and uncoupled bases is in fact a real, orthogonal matrix.

The Clebsch-Gordan coefficients have several properties which follow in a simple way from their definition. The first follows from the fact that the Clebsch-Gordan coefficients are the components of a unitary matrix, so that

$$\sum_{m_1m_2} \langle jm|j_1j_2m_1m_2\rangle\langle j_1j_2m_1m_2|j'm'\rangle = \delta_{jj'}\,\delta_{mm'}, \tag{14.30a}$$

$$\sum_{jm}\langle j_1j_2m_1m_2|jm\rangle\langle jm|j_1j_2m_1'm_2'\rangle = \delta_{m_1m_1'}\,\delta_{m_2m_2'}. \tag{14.30b}$$

Again, these are nothing but orthonormality relations for the two bases, with resolutions of the identity inserted.

Another property is the selection rule,

$$\langle jm|j_1j_2m_1m_2\rangle = 0 \quad \text{unless } m = m_1 + m_2, \tag{14.31}$$

which follows immediately from Eq. (14.21).

The Clebsch-Gordan coefficients also satisfy various recursion relations. We can obtain one of these by applying $J_- = J_{1-} + J_{2-}$ to Eq. (14.27a). This gives

$$\begin{aligned}
\frac{1}{\hbar}J_-|jm\rangle &= \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle \\
&= \sum_{m_1m_2}\Big(\sqrt{(j_1+m_1)(j_1-m_1+1)}\,|j_1j_2,m_1-1,m_2\rangle \\
&+ \sqrt{(j_2+m_2)(j_2-m_2+1)}\,|j_1j_2m_1,m_2-1\rangle\Big)\langle j_1j_2m_1m_2|jm\rangle \\
&= \sum_{m_1m_2}\Big(\sqrt{(j_1+m_1+1)(j_1-m_1)}\,\langle j_1j_2,m_1+1,m_2|jm\rangle \\
&+ \sqrt{(j_2+m_2+1)(j_2-m_2)}\,\langle j_1j_2m_1,m_2+1|jm\rangle\Big) \\
&\qquad \times |j_1j_2m_1m_2\rangle,
\end{aligned} \tag{14.32}$$

to which we apply the bra $\langle j_1 j_2 m_1' m_2'|$ from the left and rearrange indices to obtain,

$$\begin{aligned}\langle j_1 j_2 m_1 m_2 | jm\rangle &= \sqrt{(j_1+m_1+1)(j_1-m_1)}\,\langle j_1 j_2, m_1+1, m_2|jm\rangle \\ &\quad + \sqrt{(j_2+m_2+1)(j_2-m_2)}\,\langle j_1 j_2 m_1, m_2+1|jm\rangle. \end{aligned} \tag{14.33}$$

Similar recursion relations follow by using J_+, or by working with Eq. (14.27b). These recursion relations are of use in calculating the Clebsch-Gordan coefficients.

Other properties of the Clebsch-Gordan coefficients include the following identities:

$$\langle j_1 j_2 m_1 m_2 | jm\rangle = (-1)^{j_1+j_2-j}\langle j_2 j_1 m_2 m_1|jm\rangle, \tag{14.34a}$$

$$= (-1)^{j_1-j+m_2}\sqrt{\frac{2j+1}{2j_1+1}}\,\langle j j_2 m, -m_2|j_1 m_1\rangle, \tag{14.34b}$$

$$= (-1)^{j_2-j-m_1}\sqrt{\frac{2j+1}{2j_2+1}}\,\langle j_1 j, -m_1, m|j_2 m_2\rangle, \tag{14.34c}$$

$$= (-1)^{j_1+j_2-j}\langle j_1 j_2, -m_1, -m_2|j, -m\rangle. \tag{14.34d}$$

We will not prove these identities here. If you ever have to use such identities in a serious way, you should look into the Wigner $3j$-symbols, which provide a more symmetrical way of dealing with Clebsch-Gordan coefficients.

It is convenient to regard the Clebsch-Gordan coefficient $\langle jm|j_1 j_2 m_1 m_2\rangle$ as being equal to zero if any of the parameters lie outside the range for which they are meaningful, for example, if $j > j_1 + j_2$ or $m > j$.

Let us now consider the effect of the rotation operator $U(\hat{\mathbf{n}}, \theta)$ on the tensor product space $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$. This rotation operator is defined in the usual way, and can be expressed in terms of the rotation operators $U_1(\hat{\mathbf{n}}, \theta)$ and $U_2(\hat{\mathbf{n}}, \theta)$ which act on the constituent spaces $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$\begin{aligned} U(\hat{\mathbf{n}}, \theta) &= e^{-i\theta\hat{\mathbf{n}}\cdot\mathbf{J}/\hbar} = e^{-i\theta\hat{\mathbf{n}}\cdot(\mathbf{J}_1+\mathbf{J}_2)/\hbar} \\ &= e^{-i\theta\hat{\mathbf{n}}\cdot\mathbf{J}_1/\hbar} e^{-i\theta\hat{\mathbf{n}}\cdot\mathbf{J}_2/\hbar} = U_1(\hat{\mathbf{n}}, \theta) U_2(\hat{\mathbf{n}}, \theta). \end{aligned} \tag{14.35}$$

Here the exponential of the sum factors into a product of exponentials because $\mathbf{J}_1$ and $\mathbf{J}_2$ commute.

Interesting results can be obtained from this. First let us apply a rotation operator U to a vector of the uncoupled basis. We find

$$\begin{aligned} U|j_1 j_2 m_1 m_2\rangle &= (U_1|j_1 m_1\rangle)(U_2|j_2 m_2\rangle) = \sum_{m_1' m_2'} |j_1 j_2 m_1' m_2'\rangle D^{j_1}_{m_1' m_1} D^{j_2}_{m_2' m_2} \\ &= \sum_{jm}\sum_{m'} |jm'\rangle D^{j}_{m'm}\langle jm|j_1 j_2 m_1 m_2\rangle, \end{aligned} \tag{14.36}$$

where we use Eqs. (11.56) and (14.27b), and where it is understood that all D matrices have the same axis and angle $(\hat{\mathbf{n}},\theta)$. Then we multiply this on the left by $\langle j_1 j_2 m''_1 m''_2|$ and rearrange indices, to obtain,

$$D^{j_1}_{m_1 m'_1} D^{j_2}_{m_2 m'_2} = \sum_{jmm'} \langle j_1 j_2 m_1 m_2 | jm\rangle D^{j}_{mm'} \langle jm' | j_1 j_2 m'_1 m'_2\rangle. \tag{14.37}$$

In a similar manner we can obtain the identity,

$$D^{j}_{mm'} = \sum_{\substack{m_1 m_2 \\ m'_1 m'_2}} \langle jm | j_1 j_2 m_1 m_2\rangle D^{j_1}_{m_1 m'_1} D^{j_2}_{m_2 m'_2} \langle j_1 j_2 m'_1 m'_2 | jm'\rangle. \tag{14.38}$$

These identities are useful for a variety of purposes; for example, Eq. (14.37) can be used whenever it is necessary to express a product of D matrices as a linear combination of single D matrices (a problem which arises often in atomic, molecular, and nuclear physics), and Eq. (14.38) shows that D matrices for small values of j can be combined to find the D matrices for larger values of j.

We present here one application of Eq. (14.37), in which we use Eq. (12.54) to obtain an identity involving the $Y_{\ell m}$'s. First we change notation in Eq. (14.37), setting $j_1 = \ell_1$, $j_2 = \ell_2$ and $j = \ell$ (indicating integer angular momenta), and we set $m'_1 = m'_2 = 0$. Then because of the selection rule (14.31), the m' sum on the right hand side is replaced by the single term $m' = 0$. Then we take the complex conjugate of both sides and use Eq. (12.54), to obtain

$$Y_{\ell_1 m_1}(\theta,\phi) Y_{\ell_2 m_2}(\theta,\phi) = \sum_{\ell m} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \times Y_{\ell m}(\theta,\phi)\, \langle \ell 0 | \ell_1 \ell_2 00\rangle \langle \ell_1 \ell_2 m_1 m_2 | \ell m\rangle. \tag{14.39}$$

Of course, the product of two $Y_{\ell m}$'s is a function on the unit sphere, which can be expanded as a linear combination of other $Y_{\ell m}$'s; this formula gives the expansion explicitly. We see that the ℓ values which contribute are exactly those which occur in $\ell_1 \otimes \ell_2$. Finally, we can multiply this by $Y^*_{\ell_3 m_3}$ and integrate to obtain the useful formula,

$$\int d\Omega\, Y^*_{\ell_3 m_3} Y_{\ell_1 m_1} Y_{\ell_2 m_2} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}}\, \langle \ell_3 0 | \ell_1 \ell_2 00\rangle \langle \ell_1 \ell_2 m_1 m_2 | \ell_3 m_3\rangle. \tag{14.40}$$

We will call this the *three-$Y_{\ell m}$ formula*; it is very useful in atomic physics. This formula can be regarded as a special case of the Wigner-Eckart theorem, which we consider later.