A. CORREA.

PS #7

$(e \equiv 1 \quad c \equiv 1 \quad m \equiv 1)$

1) a)

$$\mathcal{L}(\bar{x}, \dot{\bar{x}}) = -\left(1 - \dot{x}_j \dot{x}_j\right)^{\frac{1}{2}} - \phi + \dot{x}_k A_k$$

EL eqns: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \qquad \forall i$

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{\partial}{\partial x_i}\phi + \dot{x}_k \frac{\partial}{\partial x_i} A_k$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = +\cancel{\tfrac{1}{2}}\left(1 - \dot{x}_j \dot{x}_j\right)^{-\frac{1}{2}} \cancel{2}\, \dot{x}_i + \frac{\partial \dot{x}_k}{\partial \dot{x}_i} A_k \qquad (*)$$

$$= \underbrace{\left(1 - \dot{x}_j \dot{x}_j\right)^{-\frac{1}{2}} \dot{x}_i}_{P_i \text{ (mechanical momentum)}} + A_i = p_i \quad \text{(canonical momentum)}$$

so

$$\frac{d}{dt}\left(p_i\right) + \frac{\partial}{\partial x_i}\phi - \dot{x}_k \frac{\partial}{\partial x_i} A_k = 0$$

$$\frac{d}{dt}\left(p_i\right) = \frac{d}{dt}\left(P_i + A_i\right) = \frac{d}{dt}\left(P_i\right) + \frac{\partial}{\partial x_\ell} A_i \dot{x}_\ell + \frac{\partial A_i}{\partial t}$$

so

$$\frac{d}{dt}\left(P_i\right) = -\frac{\partial}{\partial x_i}\phi + \dot{x}_k \frac{\partial}{\partial x_i} A_k - \frac{\partial}{\partial x_\ell} A_i \dot{x}_\ell - \frac{\partial A_i}{\partial t}$$

$$\frac{d}{dt}\left(p_i\right) = -\frac{\partial}{\partial x_i}\phi + \underbrace{\left[\frac{\partial}{\partial x_i}A_\ell - \frac{\partial}{\partial x_\ell}A_i\right]\dot{x}_\ell}_{\underbrace{\left(\delta_{ik}\delta_{\ell m} - \delta_{\ell k}\delta_{im}\right)}_{\varepsilon_{i\ell m}\varepsilon_{\ell k m}}\frac{\partial}{\partial x_k}A_m \dot{x}_\ell} - \frac{\partial A_i}{\partial t}$$

$$\boxed{\frac{d}{dt}\left(p_i\right) = -\frac{\partial}{\partial x_i}\phi + \varepsilon_{i\ell m}\,\varepsilon_{\ell k m}\frac{\partial}{\partial x_k}A_m \dot{x}_\ell - \frac{\partial A_i}{\partial t}}$$

$$\frac{d}{dt}\left(\frac{\dot{x}_i}{(1-\dot{x}_j\dot{x}_j)^{-1/2}}\right) = -\frac{\partial}{\partial x_i}\phi + \varepsilon_{i\ell m}\,\varepsilon_{\ell k m}\frac{\partial}{\partial x_k}A_m \dot{x}_\ell - \frac{\partial A_i}{\partial t}$$

in vector notation:

$$\frac{d}{dt}\left(\frac{\dot{\bar{x}}}{\sqrt{1-\dot{\bar{x}}^2}}\right) = \underbrace{-\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t}}_{\bar{E}} + \dot{\bar{x}} \times \underbrace{\left(\bar{\nabla}\times\bar{A}\right)}_{\bar{B}}$$

$$\boxed{\frac{d}{dt}\left(\frac{m\,\dot{\bar{x}}}{\sqrt{1-\dot{\bar{x}}^2/c^2}}\right) = e\left(\bar{E} + \frac{\dot{\bar{x}}}{c}\times\bar{B}\right)}$$

b)

from ⊛

$$\frac{\dot{\bar{x}}}{\sqrt{1-\dot{\bar{x}}^2}} + \bar{A} = \bar{p}$$

$$\frac{\dot{\bar{x}}}{\sqrt{1-\dot{\bar{x}}^2}} = \left(\bar{p} - \bar{A}\right)$$

$$\frac{\dot{\bar{x}}^2}{1-\dot{\bar{x}}^2} = \left(\bar{p} - \bar{A}\right)^2$$

$$\dot{\bar{x}}^2 = \left(\bar{p} - \bar{A}\right)^2\left(1 - \dot{\bar{x}}^2\right)$$

$$\dot{\bar{x}}^2\left(1 + \left(\bar{p} - \bar{A}\right)^2\right) = \left(\bar{p} - \bar{A}\right)^2$$

$$\dot{\bar{x}}^2 = \frac{\left(\bar{p} - \bar{A}\right)^2}{1 + \left(\bar{p} - \bar{A}\right)^2}$$

so

$$\boxed{\dot{\bar{x}} = \frac{\bar{p} - \bar{A}}{\sqrt{1 + \left(\bar{p} - \bar{A}\right)^2}}}$$

and

$$\sqrt{1-\dot{x}^2} = \sqrt{1 - \frac{(\bar{p}-\bar{A})^2}{1+(\bar{p}-\bar{A})^2}} =$$

$$= \sqrt{\frac{1+\cancel{(\bar{p}-\bar{A})^2} - \cancel{(\bar{p}-\bar{A})^2}}{1+(\bar{p}-\bar{A})^2}} = \frac{1}{\sqrt{1+(\bar{p}-\bar{A})^2}}$$

$$\mathcal{H} = p_i \dot{x}_i - \mathcal{L}$$

$$= \bar{p}\cdot\dot{\bar{x}} + \sqrt{1-\dot{x}^2} + \phi - \dot{x}\cdot\bar{A}$$

$$= \frac{\bar{p}\cdot(\bar{p}-A)}{\sqrt{1+(\bar{p}-\bar{A})^2}} + \frac{1}{\sqrt{1+(\bar{p}-\bar{A})^2}} + \phi - \frac{(\bar{p}-\bar{A})\cdot\bar{A}}{\sqrt{1+(\bar{p}-\bar{A})^2}}$$

$$= \frac{(\bar{p}-\bar{A})\cdot(\bar{p}-\bar{A})}{\sqrt{1+(\bar{p}-\bar{A})^2}} + \frac{1}{\sqrt{1+(\bar{p}-\bar{A})^2}} + \phi$$

$$= \frac{1+(\bar{p}-\bar{A})^2}{\sqrt{1+(\bar{p}-\bar{A})^2}} + \phi \Rightarrow \boxed{\mathcal{H} = \sqrt{1+(\bar{p}-\bar{A})^2} + \phi}$$

$$\mathcal{H} = mc^2\sqrt{1+\frac{\left(\bar{p}-\frac{e}{c}\bar{A}\right)^2}{m^2c^2}} + e\phi$$

c). eqn. of motion

$$\begin{cases} \dot{x}_i = \dfrac{\partial \mathcal{H}}{\partial p_i} \\ \\ \dot{p}_i = -\dfrac{\partial \mathcal{H}}{\partial x_i} \end{cases} \qquad \mathcal{H} = \sqrt{1+(p_j-A_j)(p_j-A_j)} + \phi$$

$$\frac{\partial \mathcal{H}}{\partial p_i} = \frac{1}{2}\Big(1+(p_j-A_j)(p_j-A_j)\Big)^{-1/2} \underbrace{\frac{\partial}{\partial p_i}(p_k-A_k)(p_k-A_k)}_{\underbrace{2\frac{\partial p_k}{\partial p_i}(p_k-A_k)}_{2(p_i-A_i)}}$$

$$\boxed{\dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \Big(1+(p_j-A_j)(p_j-A_j)\Big)^{1/2}(p_i-A_i)}$$

$$\text{so} \quad \boxed{\dot{\bar{x}} = \frac{\partial \mathcal{H}}{\partial \bar{p}} = \frac{\bar{p}-\bar{A}}{\sqrt{1+(\bar{p}-\bar{A})^2}}}$$

✓ recovering the previous result.

(***)

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i} = -\frac{1}{2}\left(1+(p_j-A_j)(p_j-A_j)\right)^{-1/2}(-2)\left[\frac{\partial}{\partial x_i}A_k\right](p_k-A_k) - \frac{\partial \phi}{\partial x_i}$$

so

$$\boxed{\dot{\bar{p}} = -\frac{\partial \mathcal{H}}{\partial \bar{x}} = \frac{\left[\bar{\nabla}\otimes\bar{A}\right](\bar{p}-\bar{A})}{\sqrt{1+(\bar{p}-\bar{A})^2}} - \bar{\nabla}\phi}$$

using (***)

$$\dot{\bar{p}} = \left[\bar{\nabla}\otimes\bar{A}\right]\dot{\bar{x}} - \bar{\nabla}\phi$$

using $\bar{P} = \bar{p} - \bar{A}$ (mechanical momentum)

$$\bar{p} = \bar{P} + \bar{A}$$

$$\dot{\bar{p}} = \dot{\bar{P}} + \frac{d\bar{A}}{dt} = \dot{\bar{P}} + \left[\bar{\nabla}\otimes\bar{A}\right]^T\dot{\bar{x}} + \frac{\partial \bar{A}}{\partial t}$$

def

$$\dot{p}_i = \dot{P}_i + \frac{\partial}{\partial x_k}A_i\,\dot{x}_k + \frac{\partial A_i}{\partial t}$$

so

$$\dot{\bar{P}} = \left[\bar{\nabla}\otimes\bar{A} - \left[\bar{\nabla}\otimes\bar{A}\right]^T\right]\dot{\bar{x}} - \bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t}$$

$$\left[\bar{\nabla}\otimes\bar{A} - \left[\bar{\nabla}\otimes\bar{A}\right]^T\right]_{ij} = \frac{\partial}{\partial x_i} A_j - \frac{\partial}{\partial x_j} A_i =$$

$$= \underbrace{\left(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}\right)}_{-\varepsilon_{ijl}\,\varepsilon_{jkl}} \frac{\partial}{\partial x_k} A_l$$

so

$$\bar{\nabla}\otimes\bar{A} - \left[\bar{\nabla}\otimes\bar{A}\right]^T = -\left[\bar{\nabla}\times\bar{A}\right]\times \cdot \qquad (\text{nice!})$$

so

$$\boxed{\dot{\bar{P}} = -\underbrace{(\bar{\nabla}\times\bar{A})}_{\bar{B}} \times \dot{\bar{X}} \underbrace{- \nabla\phi - \frac{\partial \bar{A}}{\partial t}}_{\bar{E}} \qquad \text{Lorentz force!}}$$

the other Hamiltonian equ is just the definition of canonical momentum.

2) a) $$\mathcal{L}\left(x^\mu; \frac{dx^\mu}{d\sigma}\right) = mc\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}} - \frac{eBx^1}{c}\frac{dx^2}{d\sigma}$$

<u>traslations</u>

$$x^\mu \mapsto x^\mu + \varepsilon n^\mu \qquad \text{so } Q^\mu = q^\mu + \varepsilon n^\mu$$

$$\frac{dx^\mu}{d\sigma} \mapsto \frac{dx^\mu}{d\sigma} \qquad \frac{dQ^\mu}{d\sigma} = \frac{dq^\mu}{d\sigma}$$

$$\mathcal{L}\left(\{x^\mu + \varepsilon^\mu\}; \left\{\frac{dx^\mu}{d\sigma}\right\}\right) = mc\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}} - \frac{eB}{c}\left(x^1 + \varepsilon n^1\right)\frac{dx^2}{d\sigma}$$

$$= \underbrace{mc\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}} - \frac{eB}{c}x^1\frac{dx^2}{d\sigma}} \underbrace{- \frac{eB}{c}\varepsilon n^1\frac{dx^2}{d\sigma}}$$

$$= \mathcal{L}\left(\{x^\mu\}; \left\{\frac{dx^\mu}{d\sigma}\right\}\right) + \frac{d}{d\sigma}\Lambda\left(\{x^\mu\}; \varepsilon\right)$$

where $$\boxed{\Lambda\left(\{x^\mu\}; \varepsilon\right) = -\frac{eB}{c}\varepsilon n^1 x^2}$$

so traslations are symmetries of the system ('cause $\mathcal{L} \mapsto \mathcal{L} + \frac{d}{dt}\Lambda$)

conserved quantities are

$$\sum_i F_i p_i - G$$

where

$$F^\mu(\{q\}) = \left.\frac{\partial Q(\{q\};\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = n^\mu$$

$$G(\{q\}) = \left.\frac{\partial \Lambda}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{eB}{c} n^1 x^2$$

$$p_\mu = \frac{\partial \mathcal{L}}{\partial\left(\frac{dx^\mu}{d\sigma}\right)} = \frac{mc\,\frac{dx_\mu}{d\sigma}}{\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}}} - \frac{eB}{c} x^1 \delta_\mu^{\ 2}$$

so the conserved quantity is

$$\frac{mc\,\frac{dx_\mu}{d\sigma} n^\mu}{\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}}} - \frac{eB}{c} x^1 \delta_\mu^{\ 2} n^\mu + \frac{eB}{c} n^1 x^2$$

in terms of 4-velocity

$$m\frac{dx_\mu}{d\tau} n^\mu - \frac{eB}{c} x^1 \delta_\mu^{\ 2} n^\mu + \frac{eB}{c} n^1 x^2$$

so, our conserved quantity is:

$$\boxed{m \frac{dx_\mu}{d\tau} n^\mu - \frac{eB}{c}\left(x^1 n^2 - x^2 n^1\right)}$$ for any 4-vector "n^μ"

if we set $n=(1;0;0;0) \rightarrow \boxed{m \frac{dx_0}{d\tau}}$ energy is conserved

$n=(0;1;0;0) \rightarrow$

$$m \frac{dx_1}{d\tau} + \frac{eB}{c} x^2 = \left(-m\frac{dx^1}{d\tau} + \frac{eBx^2}{c}\right)$$

mechanical momentum $\longrightarrow \boxed{\tilde{P}_x - \frac{eB}{c} y}$ is conserved

$n=(0;0;1;0) \rightarrow m \frac{dx_2}{d\tau} - \frac{eB}{c} x^1 = -\left(m\frac{dx^2}{d\tau} - \frac{eB}{c} x^1\right)$

$\boxed{\tilde{P}_y + \frac{eB}{c} x}$ is conserved

$n=(0;0;0;1) \rightarrow m\frac{dx_3}{d\tau} = -m\frac{dx^3}{d\tau}$ is conserved

$\boxed{\tilde{P}_z}$

rotations in the plane xy

$$\begin{cases} x'^0 = x^0 \\ x'^1 = x^1 - \varepsilon x^2 \\ x'^2 = x^2 + \varepsilon x^1 \\ x'^3 = x^3 \end{cases}$$

$$\mathcal{L}\left(\{x'\};\left\{\frac{dx'}{d\sigma}\right\}\right) = mc\sqrt{\frac{dx'^\mu}{d\sigma}\frac{dx'_\mu}{d\sigma}} - \frac{eB}{c}x'^1\frac{dx'^2}{d\sigma}$$

$$\frac{dx'^\mu}{d\sigma}\frac{dx'_\mu}{d\sigma} = \frac{dx'^0}{d\sigma}\frac{dx'_0}{d\sigma} + \frac{dx'^1}{d\sigma}\frac{dx'_1}{d\sigma} + \frac{dx'^2}{d\sigma}\frac{dx'_2}{d\sigma} + \frac{dx'^3}{d\sigma}\frac{dx'_3}{d\sigma}$$

$$= \frac{dx^0}{d\sigma}\frac{dx_0}{d\sigma} + \left(\frac{dx^1}{d\sigma} - \varepsilon\frac{dx^2}{d\sigma}\right)\left(\frac{dx_1}{d\sigma} - \varepsilon\frac{dx_2}{d\sigma}\right) +$$

$$+ \left(\frac{dx^2}{d\sigma} + \varepsilon\frac{dx^1}{d\sigma}\right)\left(\frac{dx_2}{d\sigma} + \varepsilon\frac{dx_1}{d\sigma}\right) + \frac{dx^3}{d\sigma}\frac{dx_3}{d\sigma}$$

$$\left(\frac{dx^1}{d\sigma} - \varepsilon\frac{dx^2}{d\sigma}\right)\left(\frac{dx_1}{d\sigma} - \varepsilon\frac{dx_2}{d\sigma}\right) = \frac{dx^1}{d\sigma}\frac{dx_1}{d\sigma} - \varepsilon\frac{dx^1}{d\sigma}\frac{dx_2}{d\sigma} - \varepsilon\frac{dx^2}{d\sigma}\frac{dx_1}{d\sigma} + \cancel{O(\varepsilon^2)}$$

$$= \frac{dx^1}{d\sigma}\frac{dx_1}{d\sigma} - 2\varepsilon\frac{dx^1}{d\sigma}\frac{dx_2}{d\sigma}$$

$$\left(\frac{dx^2}{d\sigma} + \varepsilon\frac{dx^1}{d\sigma}\right)\left(\frac{dx_2}{d\sigma} + \varepsilon\frac{dx_1}{d\sigma}\right) = \frac{dx^2 dx_2}{d\sigma\, d\sigma} + 2\varepsilon\frac{dx^1}{d\sigma}\frac{dx_2}{d\sigma}$$

so

$$\frac{dx'^{\mu}dx'_{\mu}}{d\sigma d\sigma} = \frac{dx^{\mu}}{d\sigma}\frac{dx_{\mu}}{d\sigma} - \cancel{2\varepsilon\frac{dx'}{d\sigma}\frac{dx_2}{d\sigma}} + \cancel{2\varepsilon\frac{dx'}{d\sigma}\frac{dx_2}{d\sigma}}$$

$$x'^1 \frac{dx'^2}{d\sigma} = \left(x^1 - \varepsilon x^2\right)\left(\frac{dx^2}{d\sigma} + \varepsilon\frac{dx^1}{d\sigma}\right)$$

$$= x^1\frac{dx^2}{d\sigma} - \varepsilon x^2\frac{dx^2}{d\sigma} + x^1\frac{dx^1}{d\sigma}\varepsilon - \cancel{\varepsilon^2 x^2\frac{dx^1}{d\sigma}}$$

$$= x^1\frac{dx^2}{d\sigma} + \varepsilon\left(x^1\frac{dx^1}{d\sigma} - x^2\frac{dx^2}{d\sigma}\right)$$

so

$$\mathcal{L}\left(\{x'\};\left\{\frac{dx'}{d\sigma}\right\}\right) = \mathcal{L}\left(\{x\};\left\{\frac{dx}{d\sigma}\right\}\right) \underbrace{-\frac{eB}{c}\varepsilon\left(x^1\frac{dx^1}{d\sigma} - x^2\frac{dx^2}{d\sigma}\right)}_{+\frac{d}{d\sigma}\Lambda}$$

$$\text{where}\quad \Lambda = -\frac{eB}{2c}\varepsilon\Big(\underbrace{(x^1)^2 - (x^2)^2}_{(x^2-y^2)}\Big)$$

So rotations are a symmetry

in this case

$$F^1 = \left.\frac{\partial x'^1}{\partial \varepsilon}\right|_{\varepsilon=0} = -x^2 \qquad (F^0 = F^4 = 0).$$

$$F^2 = \left.\frac{\partial x'^2}{\partial \varepsilon}\right|_{\varepsilon=0} = +x^1$$

and $$G = \frac{\partial \Lambda}{\partial \varepsilon} = -\frac{eB}{2c}\left((x^1)^2 - (x^2)^2\right)$$

$$p_\mu = \frac{\partial \mathcal{L}}{\partial\left(\frac{dx^\mu}{d\sigma}\right)} = \frac{mc\,\frac{dx_\mu}{d\sigma}}{\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}}} - \frac{eB}{c}\,x^1\,\delta_\mu{}^2$$

so the conserved quantity is

$$\underbrace{-x^2}_{F^1}\underbrace{\left[m\,\frac{dx_1}{d\tau}\right]}_{P_1} + \underbrace{x^1}_{F^2}\underbrace{\left[m\,\frac{dx_2}{d\tau} - \frac{eB}{c}\,x^1\right]}_{P_2} + \underbrace{\frac{eB}{2c}\left((x^1)^2 - (x^2)^2\right)}_{-G} =$$

$\longrightarrow$

$\longrightarrow$

$$-x^2 \tilde{p}_1 + x^1 \tilde{p}_2 - \frac{eB}{c}(x^1)^2 + \frac{eB}{2c}\left((x^1)^2 - (x^2)^2\right) =$$

$$x^1 \tilde{p}_2 - x^2 \tilde{p}_1 - \frac{eB}{2c}\left((x^1)^2 + (x^2)^2\right)$$

$$\boxed{-x\tilde{p}_y + y\tilde{p}_x - \frac{eB}{2c}\left(x^2+y^2\right)}$$

mechanical momentum (not canonical)

2) b) eqn of motion: $m\ddot{\bar{x}} = \frac{e}{c}\left(\dot{\bar{x}} \times \bar{B}\right)$

$$m\ddot{\bar{x}} = \frac{e}{c}\left(\dot{\bar{x}} \times \frac{\bar{x}}{x^3} g\right)$$

$$m\ddot{\bar{x}} = \alpha\left(\dot{\bar{x}} \times \frac{\bar{x}}{x^3}\right) \qquad \alpha = \frac{eg}{c}$$

apply "$\times \bar{x}$"

$$m\ddot{\bar{x}} \times \bar{x} = \frac{\alpha}{x^3}\left(\dot{\bar{x}} \times \bar{x}\right) \times \bar{x}$$

$$m\frac{d}{dt}\left(\dot{\bar{x}} \times \bar{x}\right) = \frac{\alpha}{x^3}\left\{\bar{x}(\bar{x}\cdot\dot{\bar{x}}) - \dot{\bar{x}}x^2\right\}$$

$$\frac{d}{dt}\left(m\dot{\bar{x}} \times \bar{x}\right) = \alpha\left\{\frac{\bar{x}(\bar{x}\cdot\dot{\bar{x}})}{x^3} - \frac{\dot{\bar{x}}}{x}\right\}$$

$$\frac{d}{dt}\left(m\dot{\bar{x}} \times \bar{x}\right) = \frac{d}{dt}\left(-\alpha\frac{\bar{x}}{x}\right)$$

$$\boxed{\frac{d}{dt}\left(\underbrace{m\dot{\bar{x}} \times \bar{x}}_{-\bar{L}_{kin.}} + \alpha\frac{\bar{x}}{x}\right) = 0}$$

sorry if too simple!

in the limit $\alpha = 0$ it is $\frac{d}{dt}\left(\bar{L}\right) = 0$ so these conserved quantities are asociated with symmetry of rotation

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The Magnetic Monopole

Another magnetic problem which has attracted a surprising amount of interest is the problem of the motion of a charged particle in the field of an isolated magnetic monopole, first analyzed by Poincare (74) in 1896. Since we no longer deal with central forces, the angular momentum is no longer conserved, and the motion is no longer necessarily planar. However, it may be thought that a certain amount of angular momentum resides in the magnetic field, and that a total angular momentum $\mathbf{D}$ exists

$$\mathbf{D} = \mathbf{L} - \epsilon\hat{\mathbf{r}}$$

in which $\mathbf{L}$ is the mechanical angular momentum $\mathbf{r} \times \mathbf{p}$, and ϵ. is the magnetic pole strength. The total angular momentum, already observed by Poincaré, is a conserved constant, and plays the role of the angular momentum when the magnetic pole strength is nonzero. In particular, where $\hat{\mathbf{r}}$ is a unit vector in the radial direction, we have

$$\hat{\mathbf{r}} \cdot \mathbf{D} = -\epsilon$$

so that the motion of a particle in the field of a magnetic monopole is always confined to the surface of a cone whose half-angle decreases with pole strength from a value of $\pi/2$ in the absence of a magnetic field. For the motion of a charged particle in the field of an uncharged monopole, the trajectory is a geodesic on the surface of the cone. The particle spirals in from infinity, is reflected, and recedes to infinity on a path asymptotic to an element of the cone.

Vectorial methods are quite adequate for treating the motion of a particle in a monopole field, and several articles using them may be found in the recent literature: Lapidus and Pietenpol (75), Nadeau (76), Lehnert (77). Quantum mechanical interest in the monopole dates from Dirac's (78) speculations of 1931 and Tamm's (79) solution of its Schrödinger equation. One of their immediate conclusions was that if the monopole were to exist as an isolated particle, its magnetic charge would have to be quantized, and as an indirect conclusion one could deduce the necessity of quantizing electric charge as well. The next two decades saw attention being paid to various details: Grönblom (80) and Jordan (81) examined how critical the location of the singularity in the vector potential might be for the solutions, Saha (82) speculated whether a magnetic charge might account for the proton's greater mass than the electron, while Wilson (83), Eldridge (84), Saha (85), and Ramsey (86) made miscellaneous contributions or suggestions. Of more importance were Fierz' (87) alternative derivation of Dirac's and Tamm's results, where in particular the role of Poincaré's vector $\mathbf{D}$ as the generator of infinitesimal rotations and the angular momentum of the field were given explicit treatment, and Banderet's (88) treatment of scattering from a monopole. Ford and Wheeler (89) made a semiclassical analysis of scattering from a monopole; Goto (90) speculated on their behavior in cosmic space, and Wentzl (91) wrote a short note on their properties. Harish-Chandra (92), solving the problem relativistically with Dirac's equation, found that not even the dipole magnetic moment of the electron would lead to a bound state with the monopole, a result which did not modify earlier conclusions about a non-spinning electron obtained from the solutions of Schrödinger's equation, The properties of an electrically charged magnetic monopole were investigated by Eliezer and Roy (93) using Schrödinger's equation, while Malkus (94) investigated the energy levels of a charged monopole using Pauli's approximation. Unfortunately the more exact treatment with the Dirac equation leads to a singular potential in the lowest angular momentum states (95).

With the exception of the last three papers in which an explicit electric attraction was included, no bound states were ever found, and so again questions of symmetry were relatively unimportant. Moreover, the magnetic field removes the accidental degeneracy of the Coulomb problem, so that the various authors would have had no reason to have been concerned with symmetries or degeneracy anomalies either. Since angular momentum is no longer conserved, rotational symmetry must be given a cautious treatment, but it would seem that only Fierz gave the matter much attention.

There is an appreciable number of experimental papers, concerned with continuing efforts to detect isolated monopoles, which to date have been uniformly negative. Additionally during the past decade there have been a number of papers treating monopoles from the point of view of quantum field theory. They have two motives: one to obtain additional information which might assist the efforts toward experimental detection, and the other to determine whether there would be any inconsistency or contradiction in field theory itself which might rule out the existence of monopoles. Some of the papers are noteworthy from the point of view of symmetry and degeneracy, particularly Schwinger's (96) arguments that the quantized pole strength ought to be twice that originally required by Dirac, and Peres (97) confirmation of this result from symmetry principles, which depend somewhat upon assumptions regarding the properties of angular momentum. Another potentially interesting paper (98) treats the angular momentum of the field from such an extrinsic point of view that it is hard to reconcile with the known symmetry properties.

In point of fact, symmetry in the presence of a magnetic field requires rather careful attention. Even though the magnetic field may possess a certain symmetry, such as translational invariance or spherical symmetry, the vector potential from which it is derived will more than likely not possess the same symmetry. A symmetry operation will therefore result in a gauge transformation. Since, in Hamiltonian mechanics, the vector potential is to be added to the canonical momentum, any change in the vector potential produced by a symmetry operation will be manifested as a gauge, which will have to be canceled in the canonical transformation which is applied to the momentum operator. Therefore the generators of an infinitesimal symmetry transformation which one would expect in the absence of a magnetic field have to be modified by the infinitesimal gauge transformation which that symmetry operation produces. In just this way Poincaré's vector $\mathbf{D}$ replaces the angular momentum for the spherically symmetrical monopole field, and in fact similar considerations can be seen to apply to the problem of cyclotron motion.

For the uncharged monopole, there is perhaps very little more to be said. For a charged monopole, the accidental degeneracy of the Coulomb field is lost, but it may be recovered in a way whose mathematical elegance outweighs its physical artificiality. This way is to add a repulsive centrifugal potential roportional to the square of the magnetic pole strength, whereupon both the Coulomb problem and the harmonic oscillator exhibit accidental degeneracy. The mechanism may be understood if we recall that the effect of a centrifugal potential is to cause an orbital precession. In the case of a central force the precession occurs in the plane of motion, and in the present case it takes place about the vector of total angular momentum. The most noticeable difference between central force motion with and without the magnetic field of the monopole is its confinement to the surface of Poincare's cone, so that motion which would ordinarily be planar is simply rolled up on the surface of the cone. However, an orbit which would be closed in the plane will extend around more than one circumference on the cone. This effect can be counteracted by an appropriate precession.

The attempted recovery of our two classically degenerate systems in this manner leaves the harmonic oscillator still somewhat intractable, but the Coulomb potential acquires some quite surprising attributes. First of all, it is possible to construct a Runge vector based on the total angular momentum rather than on the mechanical angular momentum. It is no longer orthogonal to the angular momentum, either mechanical or total. It may be shown that the orbit is again planar, although our considerations only led us to expect that it would be closed. The monopole no longer occupies the focus of the conic section comprising the orbit, nor does it even lie in the plane of the orbit. Nevertheless, the Runge vector and the total angular momentum still generate an $O(4)$ group of constants of the motion, classically as well as quantum mechanically, a group which is adequate to account for the degeneracy which is observed. Here there is something of a surprise, for the minimum angular momentum value which may occur is that determined by the quantized magnetic pole strength. Moreover, the irreducible representations which occur are not those of dimension n^2 characteristic of orbital angular momentum, but rather those of dimension mn, where $m - n = 2\epsilon$. If one wishes to avoid wave functions transforming according to half-integral representations of the rotation group, and therefore which would be double valued, he is forced to accept Schwinger's value for the minimum quantized pole strength, rather than half that amount, which was Dirac's original quantum. This result depends to a considerable extent on the choice of gauge. Finally, even though the symmetry group is $O(4)$, some such transformation as Fock's to force-free motion on a hypersphere is ruled out, since the mn representations do not exist for orbital angular momentum.

As the folklore requires, our charged monopole is separable in a variety of coordinate systems, including polar, parabolic, quaternionic, and ellipsoidal. This latter system is interesting, inasmuch as it also allows the separation of the equation for a pair of charged monopoles with the repulsive centrifugal potential. A detailed treatment of the symmetry of the monopole problems was given by McIntosh and Cisneros (99a). Two papers by Zwanziger (99b) also contain a wealth of detail, particularly regarding magnetic sources in the Maxwell equations, scattering from the monopole, and separation of the relative coordinates for the two-body problem.

Although the problem of classical motion in the field of a magnetic dipole is of considerable practical and theoretical interest, it is a vastly more complicated system to treat, starting with the nonseparability of the radial and colatitudinal coordinates in its Hamilton-Jacobi equation. Extensive work on classical motion in the field of a magnetic dipole, also including an attractive Coulomb potential, has been carried out by Störmer (100). The motion of cosmic rays in the magnetic field of the earth is thought to follow such a model, at least as an approximation. Quantum mechanically, such a system is of interest for its application to the scattering of neutrons, whose principal mode of interaction is through their magnetic dipole moment.

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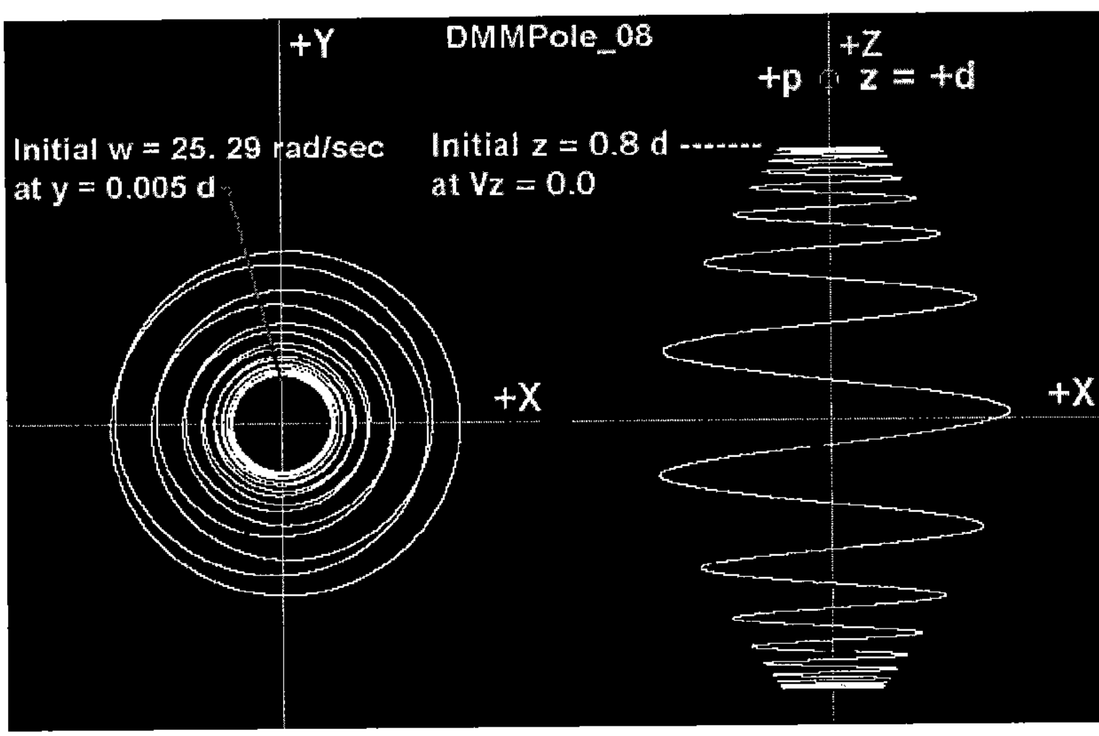
Electron Orbits in Three Dimensions

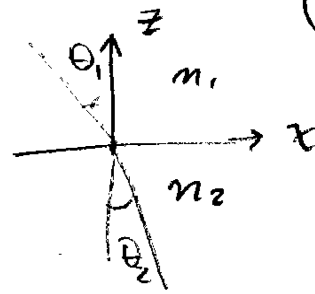
In the two figures below, we show, for the first time, classical electron motion along the z-axis between two appositely charged magnetic monopoles (+p and -p) located at z = +d and z = -d. Since this motion appears to be unstable for larger radii, the xy-plane scale has been magnified to accommodate the view for the smaller orbits. In the first figure, the electron is initially given a circular orbit (w = 25.29 rad/sec at r = 0.005 d), centered upon the z-axis, with zero velocity along that axis, at a distance of 0.2 d from the positive magnetic pole. This circular motion causes the electron orbit, as a whole, to be repulsed by the increasing magnetic field gradient from the nearby magnetic monopole. The orbit, increasing in radius to enclose constant magnetic flux, trading orbital velocity for linear motion along the axis, accelerates away from the nearby pole. After midpoint between the two poles, the electron orbit is repulsed by the nearer negative pole field gradient, and mirrors its previous trajectory, until it ends with it's initial small, rapidly rotating orbit , a distance, 0.2 d, away from the negative magnetic monopole. The electron orbit will then reverse it's linear motion, and start it's journey back towards the positive pole to finish the first cycle of periodic motion between the two poles.

The second figure has the same initial conditions as the first, except that, the angular velocity is now 1/4 of that shown previously. This causes circular motion that is off axis from the line connecting the two poles. [Old Man]



3) $P[\bar{x}(\lambda)] = \int_{\lambda_0}^{\lambda_1} n(\bar{x}) \left|\frac{d\bar{x}}{d\lambda}\right| d\lambda$

is like $\mathcal{L}(\bar{x};\dot{\bar{x}}) = n(\bar{x})|\dot{\bar{x}}|$ $\quad n(\bar{x}) = n(z) = \begin{cases} n_2 & z<0 \\ n_1 & z>0 \end{cases}$



$\mathcal{L} = n(z)\sqrt{\dot{x}^2+\dot{z}^2}$

x: cyclic variable $\Rightarrow$ $\boxed{P_x \text{ is conserved}}$

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = n(z)\frac{1}{\cancel{2}}\frac{\cancel{2}\dot{x}}{\sqrt{\dot{x}^2+\dot{z}^2}}$$

then (if ray pass thru $z=0$) $n_1 \frac{dx_1}{\sqrt{dx_1^2+dz_1^2}} = n_2 \frac{dx_2}{\sqrt{dx_2^2+dz_2^2}}$

so $\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$ Snell Law

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as λ does not appear in $\mathcal{L}$ there is another conserved quantity $\mathcal{H} = \sum_i p_i \dot{x}_i - \mathcal{L}$

$$\mathcal{H} = P_x\dot{x} + P_z\dot{z} - \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\dot{x}}\dot{x} + \frac{\partial\mathcal{L}}{\partial\dot{z}}\dot{z} - \mathcal{L} =$$

$$= n(z)\left\{\frac{\dot{x}^2+\dot{z}^2}{\sqrt{\dot{x}^2+\dot{z}^2}} - \sqrt{\dot{x}^2+\dot{z}^2}\right\} = 0$$

not additional information!

4) $$L = \sum_i^N \frac{m}{2}\dot{y}_i^2 - \sum_{i=1}^{N+1} \frac{k}{2}(y_i - y_{i-1})^2$$

L is quadratic in velocities so

$$H = \sum_i^N \frac{p_i^2}{2m} + \sum_{i=1}^{N+1} \frac{k}{2}(y_i - y_{i-1})^2$$

(where $p_i = m\dot{y}_i$)

the limit $N \to \infty$; $m = \rho\Delta x$; $k = \kappa/\Delta x$ completelly analogous as in the case of the Lagrangian; so

$$H = \int \mathcal{H}\, dx$$

where $$\boxed{\mathcal{H} = \frac{\pi^2}{2\rho} + \frac{\kappa}{2}\left(\frac{\partial y}{\partial x}\right)^2}$$

(where $\pi(x) = \rho \dfrac{\partial y}{\partial t}(x)$)

~~eqn of motion~~

$$\cancel{\left\{\begin{array}{l} \dfrac{\partial y}{\partial t}(x) = \dfrac{\partial H}{\partial \pi(x)} = \dfrac{\pi(x)}{\rho} \\ \dfrac{\partial \pi(x)}{\partial t} = -\dfrac{\partial H}{\partial y(x)} = -\kappa\dfrac{\partial y}{\partial x}(x) \end{array}\right.}$$

in terms of y and derivatives

$$\boxed{\mathcal{H} = \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{K}{2}\left(\frac{\partial y}{\partial x}\right)^2}$$

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\rho}{\cancel{2}} \cancel{2} \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} + \frac{K}{\cancel{2}} \cancel{2} \left(\frac{\partial y}{\partial x}\right) \frac{\partial^2 y}{\partial x \partial t}$$

$$= \rho \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} + K\left(\frac{\partial y}{\partial x}\right) \frac{\partial^2 y}{\partial x \partial t}$$

using the eqn of motion $\rho \frac{\partial^2 y}{\partial t^2} - K \frac{\partial^2 y}{\partial x^2} = 0$

$$\frac{\partial \mathcal{H}}{\partial t} = K \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2} + K \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t}$$

$$= \frac{\partial}{\partial x}\left(K \frac{\partial y}{\partial t} \frac{\partial y}{\partial x}\right)$$ so

$$\boxed{\frac{\partial \mathcal{H}}{\partial t} + \frac{\partial}{\partial x}\left(-K \frac{\partial y}{\partial t} \frac{\partial y}{\partial x}\right) = 0}$$

$$\boxed{S = -K \frac{\partial y}{\partial t} \cdot \frac{\partial y}{\partial x}}$$

S is interpreted as a flux of energy