PS # 8 prepared by
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1) a)

$$\frac{d}{d\tau}(s \cdot u) = u_\mu \underbrace{\frac{dS^\mu}{d\tau}}_{\downarrow \text{ BMT eqn.}} + S_\mu \underbrace{\frac{du^\mu}{d\tau}}_{\downarrow \text{ relativ. force}}$$

$$\frac{dS^\mu}{d\tau} = \frac{e}{mc}\left[\frac{g}{2} F^\mu{}_\nu S^\nu - \frac{1}{c^2}\left(\frac{g}{2}-1\right)(u^\alpha F_{\alpha\beta} S^\beta)\, u^\mu\right]$$

$$\frac{du^\mu}{d\tau} = F^\mu{}_\nu u^\nu \frac{e}{mc}$$

$$\downarrow$$

$$\frac{d}{d\tau}(s \cdot u) = \frac{e}{mc}\left[\cancel{\frac{g}{2}} u_\mu F^\mu{}_\nu S^\nu - \cancel{\frac{1}{c^2}}\left(\cancel{\frac{g}{2}}-1\right)(u^\alpha F_{\alpha\beta} S^\beta)\, \overbrace{\cancel{u_\mu u^\mu}}^{c^2}\right]$$

$$+ S_\mu F^\mu{}_\nu u^\nu$$

$$= \frac{e}{mc}\left[+1 \cdot u^\alpha F_{\alpha\beta} S^\beta\right] + S_\mu F^\mu{}_\nu u^\nu \frac{e}{mc}$$

$$= \frac{e}{mc}\left[u^\alpha F_{\alpha\beta} S^\beta + \underbrace{u^\nu F^\mu{}_\nu S_\mu}_{-u^\nu F_\nu{}^\mu S_\mu}\right] \equiv \emptyset \quad \checkmark$$

if at one time $(s \cdot u) = 0 \Rightarrow (s \cdot u) = 0$ for all times

1) a) cont'd

$$\frac{d}{d\tau}(p\cdot q) = \left(p\cdot\frac{dq}{d\tau}\right) + \left(q\cdot\frac{dp}{d\tau}\right)$$

$$= p_\mu \underbrace{\frac{dq^\mu}{d\tau}}_{\text{BMT}} + q_\mu \underbrace{\frac{dp^\mu}{d\tau}}_{\text{BMT equ.}}$$

since $p_\mu u^\mu = \emptyset$

$$= p_\mu\left[\frac{g}{2}F^\mu{}_\nu q^\nu - \cancel{\frac{1}{c^2}\left(\frac{g}{2}-1\right)(u\cdot F\cdot q)\,u^\mu}\right] + \quad \emptyset$$

$\emptyset$ since $q_\mu u^\mu = \emptyset$

$$+\, q_\mu\left[\frac{g}{2}F^\mu{}_\nu p^\nu - \cancel{\frac{1}{c^2}\left(\frac{g}{2}-1\right)(u\cdot F\cdot p)\,u^\mu}\right]$$

$$= \frac{g}{2}\left[p_\mu F^\mu{}_\nu q^\nu + \underbrace{\underbrace{q_\mu F^\mu{}_\nu p^\nu}_{q_\nu F^\nu{}_\mu p^\mu}}_{q^\nu F_\nu{}^\mu p_\mu}\right]$$

$$= \frac{g}{2}\left[p_\mu\underbrace{\left\{F^\mu{}_\nu + F_\nu{}^\mu\right\}}_{=\emptyset}q^\nu\right] = \emptyset.$$

since F is antisymmetric.

1) b) $\quad S^i = -\left(\hat{e}_i \cdot \bar{S}\right)$

$$\frac{dS^i}{d\tau} = -\hat{e}_i \cdot \frac{d\bar{S}}{d\tau} - \frac{d\hat{e}_i}{d\tau} \cdot \bar{S}$$

$$-\frac{d\hat{e}_i}{d\tau} \cdot \vec{S} = \left(\vec{\omega}_T \times \vec{S}\right)_i \quad \text{where} \quad \vec{\omega}_T = \frac{\gamma^2}{1+\gamma} \frac{\vec{a} \times \vec{v}}{c^2}$$

$$-\hat{e}_i \cdot \frac{dS}{d\tau} = -\left(\frac{dS}{d\tau}\right)^i = -\frac{e}{mc}\left[\frac{g}{2} F^i{}_\nu S^\nu - \frac{1}{c^2}\left(\frac{g}{2}-1\right)(u \cdot F \cdot S)u^i\right]$$

$u^i = 0$

$i = 1; 2; 3.$

F contains the magnetic field.

$$= -\frac{e}{mc}\frac{g}{2}\left(\bar{B} \times \bar{S}\right)_i = \frac{e}{mc}\frac{g}{2}\left(\bar{B} \times \bar{S}\right)^i$$

so:

$$\frac{dS_i}{d\tau} = \left(\vec{\omega}_T \times \vec{S}\right)_i - \frac{e}{mc}\frac{g}{2}\left(\bar{B} \times \bar{S}\right)_i$$

$$\frac{dS_i}{d\tau} = \left[\left(\vec{\omega}_T - \frac{e}{mc}\frac{g}{2}\bar{B}\right) \times \bar{S}\right]_i$$

$$\frac{d}{dt} = \gamma \frac{d}{d\tau}$$

$$\bar{B}' = \gamma \bar{B}$$

$\downarrow$

$$\frac{dS_i}{dt} = \frac{1}{\gamma}\left[\left(\vec{\omega}_T - \frac{e}{mc}\frac{g}{2}\gamma\bar{B}\right) \times \bar{S}\right]_i$$

1) c)

Precession rate in Thomas rest frame:

$$\frac{\gamma e B}{mc}\left[\frac{\gamma}{(\gamma+1)}\frac{v^2}{c^2} - \frac{g}{2}\right]$$

in the Fomas frame the period is

$$\frac{2\pi\gamma}{(eB/mc)}\frac{1}{\gamma c}$$

multipliing we get the angle of precession

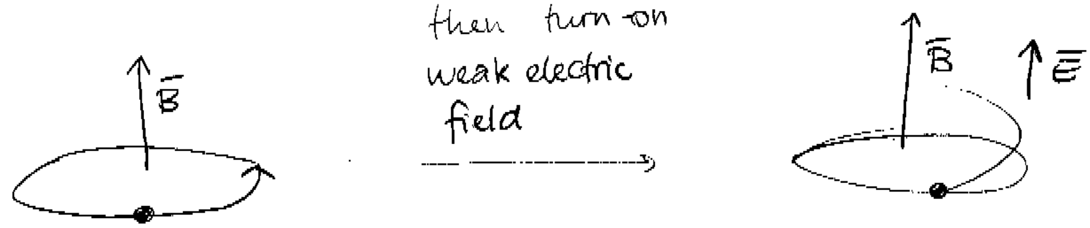
$$\Theta = \frac{2\pi\gamma}{c}\left[\frac{\gamma\beta^2}{(\gamma+1)} - \frac{g}{2}\right]$$

$$= 2\pi\left[\gamma - \gamma\frac{g}{2} - 1\right] = \gamma 2\pi a - 2\pi$$

anomalous precession

$$\left(1-\frac{g}{2}\right)$$

2)



in $\hat{z}$ direction

$$\frac{d}{dt} P_z = eE$$

$$\frac{d}{dt}\left(\gamma m v_z\right) = eE \qquad \text{equation of motion}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \doteq \frac{1}{\sqrt{1-\frac{v_\perp^2+v_z^2}{c^2}}} \sim \frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}$$

v_z is very small.
an γ does not
depend on t

$$\frac{d}{dt}\left(\frac{m v_z}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\right) = eE$$

$$\underbrace{\frac{m}{\sqrt{1-\frac{v_\perp^2}{c^2}}}}\frac{d}{dt} v_z = eE$$

effective inertial mass seen by electric field.
and is the total (rest + Kinetic) energy
of the perpendicular movement.
show this for small $v_\perp$: $\frac{m}{\sqrt{1-v_\perp^2/c^2}} = mc^2 + \frac{m v_\perp^2}{2}$?

3) $x^\mu = \begin{pmatrix} ct \\ \bar{x} \end{pmatrix} \mapsto u^\mu = \begin{pmatrix} c\gamma \\ \gamma \bar{u} \end{pmatrix}$

$$\begin{cases} mc \dfrac{du^0}{d\tau} = e\bar{E}\cdot\vec{u} \\ mc \dfrac{d\bar{u}}{d\tau} = e\left(u^0\bar{E} + \bar{u}\times\bar{B}\right) \end{cases}$$

$$\begin{cases} (a) & mc \dfrac{du^0}{d\tau} = eE_0 u^1 \cos\phi \\ (b) & mc \dfrac{du^1}{d\tau} = eE_0 (u^0 - u^3)\cos\phi \\ (c) & mc \dfrac{du^3}{d\tau} = eE_0 u^1 \cos\phi \end{cases} \longrightarrow mc \frac{d(u^0 - u^3)}{d\tau} = 0$$

$$\Downarrow$$

$$u^0 - u^3 = \text{constant} = A$$

$$\Downarrow$$

$$x^0 - x^3 = A\tau$$

note that:

$$\phi = \omega t - kz = ckt - kz = k(ct - z) = k(x^0 - x^3) = kA\tau$$

pluging the result in (b) $\boxed{mc\dfrac{du_1}{d\tau} = eE_0 A \cos(kA\tau)}$

so $mc\, u_1 = \dfrac{eE_0 \cancel{A} \sin(kA\tau)}{k\cancel{A}} \rightarrow \boxed{mc u^1 = eE_0 \sin(kA\tau)/k}$

so (c) $\Rightarrow$

$$mc\frac{du^3}{d\tau} = \cancel{eE_0}(eE_0)^2 \sin(kA\tau)/kmc \times \cos(kA\tau)$$

$$\frac{du^3}{d\tau} = \frac{(eE_0)^2}{k m^2 c^2} \sin(kA\tau)\cos(kA\tau)$$

$$\downarrow$$

$$u^3 = \frac{(eE_0)^2}{k m^2 c^2} \frac{\sin^2(kA\tau)}{2kA} = \frac{(eE_0)^2}{4k^2 A m^2 c^2}\left(1-\cos 2kA\tau\right)$$

$$u^3 = u^0 - A$$

at $t=0$

$$\begin{cases} u^0 = c \\ u^3 = 0 \end{cases}$$

$$\Downarrow$$

$$A = c$$

$$u^3 = \frac{(eE_0)^2}{4\underbrace{k^2c^2}_{\omega^2} m^2 c}\left(1-\cos 2\underbrace{kc}_{\omega}\tau\right)$$

$$u^0 = \underbrace{\left[\frac{(eE_0)^2}{m^2c^2}\right]}_{\omega_0^2}\frac{c}{4\omega^2}\left(1-\cos 2\omega\tau\right) + c$$

lab frame

$$\boxed{u^3 = \frac{\omega_0^2 c}{4\omega^2}\left(1-\cos 2\omega\tau\right)} = \gamma v^{(3)}$$

given by

$$u^0 = \frac{\omega_0^2 c}{4\omega^2}\left(1-\cos 2\omega\tau\right) + c = c\gamma \Rightarrow$$

$$\boxed{v^{(3)} = \frac{\frac{\omega_0^2 c}{4\omega^2}\left(1-\cos 2\omega\tau\right)}{\frac{\omega_0^2 c}{4\omega^2}\left(1-\cos 2\omega\tau\right)+c}\, c}$$

time average

$$\boxed{\langle v^{(3)}\rangle = c\,\frac{\omega_0^2}{\omega_0^2+4\omega^2}}$$

if $m \to 0$ $\langle v^{(3)}\rangle = c$

but τ is not defined.

4) $\boxed{\mathcal{L} = \dot{x}^2 - x^2}$

$x = x^* + \varepsilon$ with $\varepsilon(0) = \varepsilon(T) = \emptyset$ $\quad \ddot{x}^* = -x^*$

$\dot{x} = \dot{x}^* + \dot{\varepsilon}$ and x^* a solution of Euler $\mathcal{L}$ eqn

$$\delta S = \int_0^T \underbrace{\mathcal{L}(x^* + \varepsilon;\ \dot{x}^* + \dot{\varepsilon})}_{\dot{x}^{*2} + 2\dot{x}^*\dot{\varepsilon} + \dot{\varepsilon}^2 - x^{*2} - 2x^*\varepsilon - \varepsilon^2} dt - \int_0^T \underbrace{\mathcal{L}(x^*;\ \dot{x}^*)}_{\dot{x}^{*2} - x^{*2}} dt$$

$$\delta S = \int_0^T 2\dot{x}^*\dot{\varepsilon} + \dot{\varepsilon}^2 - 2x^*\varepsilon - \varepsilon^2 \, dt$$

$$\int_0^T \dot{x}^*\dot{\varepsilon}\, dt \underset{\text{by parts}}{=} \dot{x}\varepsilon \Big|_0^T - \int \ddot{x}^* \varepsilon \, dt$$

$$\delta S = \int_0^T (-)2\underbrace{[x^* + \ddot{x}^*]}_{=\emptyset}\varepsilon + \dot{\varepsilon}^2 - \varepsilon^2 \, dt = \int_0^T \dot{\varepsilon}^2 - \varepsilon^2 dt$$

$$\boxed{\delta S = \int_0^T (\dot{\varepsilon}^2 - \varepsilon^2)\, dt}$$

is like anharmonic oscillator with easier constrain conditions ($\varepsilon(0) = \varepsilon(T) = \emptyset$)

consider $\varepsilon = \sum_{n=1}^{\infty} a_n \sin\left(n\pi \frac{t}{T}\right)$ complete set that automatically gives $\varepsilon(0) = \varepsilon(T) = \emptyset$

$$\dot{\varepsilon} = \sum_{n=1}^{\infty} a_n \frac{n\pi}{T} \cos\left(n\pi \frac{t}{T}\right)$$

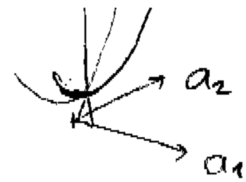
by Parseval: $\int_0^T \varepsilon^2 dt = \sum_{n=1}^{\infty} a_n^2$

and

$$\int_0^T \dot{\varepsilon}^2 dt = \sum_{n=1} \left(a_n \frac{n\pi}{T}\right)^2$$

so

$$\boxed{\delta S = \sum_{n=1}^{\infty} \left(a_n \frac{n\pi}{T}\right)^2 - a_n^2}$$



$$= \sum_{n=1}^{\infty} \left(\left[\frac{n\pi}{T}\right]^2 - 1 \right) a_n^2 \quad \checkmark$$

so $\varepsilon \equiv \emptyset$ ($a_n = \emptyset$) is a minimum ONLY

if $\left[\frac{n\pi}{T}\right]^2 > 1 \quad \forall n$

i.e. $\boxed{\pi > T}$ (in conventional units) $\left(\frac{2\pi}{\omega} > T\right)$

if $T > \pi$ then it is a saddle point

with $\mathrm{Int}\left(\frac{T}{\pi}\right)$ directions in which action reduces

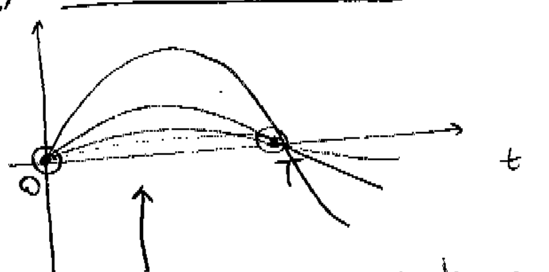
so for $T \neq n\pi \quad n \in \mathbb{N}$

all a_n must be $\emptyset$

the Hamilton principle is not well defined for $T = n\pi$

aparition of 1st saddle point:

$\varepsilon(t)$



infinite set of solutions (a_1 is arbitrary)!