PS#7.

Alfredo CORREA

1) a) 4-momentum conservation

initially $P^i = \begin{pmatrix} m_0 c \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ← energy/c; 0-velocity

at some point of trip

$$p^f = \begin{pmatrix} \gamma m c \\ \gamma m v \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E/c \\ -E/c \\ 0 \\ 0 \end{pmatrix}$$

← photons exhausted in $-\hat{x}$ direction

↑ 4-momentum of ship w/velocity v an rest mass m (but $m \neq m_0$ because of annihilation of matter-antimatter)

↑ 4-momentum of (all) photons exhausted.

so

$$\begin{cases} \gamma m c + E/c = m_0 c \\ \gamma m v - E/c = 0 \end{cases}$$

eliminating E/c

$$\gamma m (c + v) = m_0 c$$

$$\frac{m}{m_0} = \frac{c}{\gamma (c+v)}$$

$$\boxed{\frac{m}{m_0} = \frac{1}{\gamma(1 + v/c)} = \sqrt{\frac{1-\beta}{1+\beta}}}$$ (TM)

b) $a = \frac{a'}{\gamma^3} = \frac{g}{\gamma^3} \leftarrow$

$$\frac{dv}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{3/2} g \qquad \frac{dv}{dx} = \frac{dv}{dt}\frac{dt}{dx} = \frac{dv}{dt}/v$$

$$\frac{dv}{dx} = \left(1 - \frac{v^2}{c^2}\right)^{3/2} v g$$

$$\int_{v(x=0)}^{v(x)} v'\left(1 - v'^2/c^2\right)^{3/2} dv' = \int_0^x g\, dx'$$

$$\cancel{\tfrac{1}{2}} c^2 \left(1 - v'^2/c^2\right)^{-1/2}\Big|_{v(x=0)}^{v(x)} = xg$$

$$\left(1 - \frac{v^2(x)}{c^2}\right)^{-1/2} - 1 = \frac{xg}{c^2}$$

$$\frac{v^2(x)}{c^2} = 1 - \left[\frac{xg}{c^2} + 1\right]^{-2}$$

$$\boxed{v(x) = c\sqrt{1 - \left[\frac{xg}{c^2} + 1\right]^{-2}}} = c\sqrt{1 - \left(\frac{c^2}{gx + c^2}\right)^2}$$

replace $x = 2$ light year $\rightarrow \Delta v = 0.94\,c \rightarrow \frac{m}{m_0} = 0.14$

in each quarter of round trip

$$\boxed{\left(\frac{m}{m_0}\right)_{TOTAL} = (0.14)^4 \sim 8 \times 10^{-4}}$$

c) – From Earth: since we already know v_{max} it is easier to start w/: $\frac{dv}{dt} = g\gamma^{-3}$

$$\Downarrow$$

$$dv = g\gamma^{-3}\,dt$$

$$\gamma^3 dv = g\,dt$$

$$\int^{v_f} \gamma^3 dv = g\int^{T} dt$$

$$\gamma^3 = \left[\sqrt{1 - v^2/c^2}\right]^{-3/2}$$

$$T = \frac{v_f}{\sqrt{1 - v_f^2}} \longrightarrow T = 2.8 \text{ years}$$

replacing $v_f \to 0.94c$ & multipliing by 4

total time seen from Earth = $\boxed{11 \text{ years}}$

– From ship: $\gamma^3 dv = g\,dt$ and $dt = \gamma\,d\tau$

so $\gamma^2 dv = g\,d\tau$

$$\int^{v_f} \frac{dv}{1 - v^2/c^2} = \int^{\Theta} g\,d\tau$$

$$\Theta = \frac{c}{g}\,\frac{1}{2}\ln\left(\frac{1+v_f}{1-v_f}\right) \ldots \text{by } 4 \rightarrow \boxed{6.8 \text{ years}}$$

2) a) current-density form a 4-vector

$$\begin{pmatrix} n \\ J_x = \rho v_x = n c \cos\theta \\ J_y = \rho v_y = n c \sin\theta \\ 0 \end{pmatrix}$$

↓ Lorentz transform.

$$n' = \gamma\left(n - \frac{v}{c^2} J_x\right)$$

$$n' = \gamma n \left(1 - \frac{v}{c}\cos\theta\right)$$

$$J_x' = \gamma\left(J_x - \rho n\right)$$

$$J_y' = J_y$$

$$\begin{pmatrix} \gamma n\left(1 - \frac{v}{c}\cos\theta\right) \\ \gamma\left(J_x - \rho n\right) \\ J_y \\ 0 \end{pmatrix}$$ in primed frame

the angle of propagation in primed frame is

$$\tan\theta' = \frac{J_y'}{J_x'} = \frac{\sin\theta}{\gamma\left(\cos\theta - \frac{v}{c}\right)} = \frac{k_y'}{k_x'}$$

↑ Consistant w/ law of abberation

2) b)

$$u = \hbar \omega n$$

$$u' = \hbar \omega' n'$$

$$\begin{cases} \omega' = \gamma[\omega + \beta k \cos\theta] \\ n' = \gamma[n - \beta k \cos\theta] \end{cases}$$

$$\Downarrow$$

$$u' = u \gamma^2 \left(1 + \frac{v}{c}\cos\theta\right)^2$$

$\uparrow$ not a time component transformation of a 4-vector!

conclussion:
ω is time component of some 4-vector
n " " " " " "
(ωn) cannot be a component of any 4-vector
(but can be a component of a 4-tensor)

c) Brightness:

$$B_0 = c\,u\,\frac{dN}{d\Omega} \qquad B_{tot} = \int B_0\, d\Omega = \underset{\uparrow \atop \text{isotropic}}{4\pi B_0}$$

$$B_0' = u'c\,\frac{dN'}{d\Omega'} = c\,u\,\gamma^2\left(1-\frac{v}{c}\cos\theta\right)^2\frac{dN'}{d\Omega'}$$

$$dN = dN' \ (\text{invariant})$$

$$B_0' = \underbrace{\boxed{c\,u}\,\gamma^2(1-\beta\cos\theta)^2\,\frac{dN}{d\Omega}}_{B_0}\,\frac{d\Omega}{d\Omega'}$$

$$B_0' = B_0\,\frac{d\Omega}{d\Omega'}\,\gamma^2(1-\beta\cos\theta)^2$$

$$\int B_0'\,d\Omega' = \int B_0\,d\Omega\,\gamma^2(1-\beta\cos\theta)^2$$

$$\boxed{\int B_0'\,d\Omega' = \gamma^2\,4\pi B_0\left[1+\frac{1}{3}\beta^2\right]}$$

3) a) primed frame

$$\begin{cases} \bar{B}' = \emptyset \\ \bar{E}' = q\dfrac{\bar{r}'}{r'^3} = \left(\dfrac{qx'}{r'^3} ; \dfrac{qy'}{r'^3} ; q\dfrac{z'}{r'^3} \right) \end{cases}$$

transformation of fields:

$$\begin{cases} E_x = \gamma\left(E_x' + v\cancel{B_y'}\right) + \cancel{\frac{\gamma^2}{\gamma+1} v_x (v\cdot E')} \\ E_y = \gamma\left(E_y' - \cancel{\beta B_x}\right) + \cancel{v_y (v\cdot E')} \\ E_z = E_z' \end{cases} \Rightarrow \begin{cases} B_x = \gamma\left(\cancel{B_x'} - vE_y'\right) = -vE_y \\ B_y = \gamma\left(\cancel{B_y'} + vE_x'\right) = vE_x \\ B_z = \cancel{B_z'} \end{cases}$$

and also the arguments of the fields must be transformed

$$\begin{cases} \dfrac{x' \text{ or } y'}{r'^3} = \dfrac{x \text{ or } y}{\underbrace{\left[x^2 + y^2 + \gamma^2(z - vt)^2\right]^{3/2}}_{\gamma^3 R_v^3}} \\ \dfrac{z'}{r'^3} = \dfrac{(z - vt)}{\underbrace{\left[x^2 + y^2 + \gamma^2(z - vt)^2\right]^{3/2}}_{\gamma^3 R_v^3}} \end{cases}$$

so

$$E_x = qx \frac{1}{\gamma^2 R_v^3} \xrightarrow[v \to 1]{} q\frac{x}{r_\perp^2} 2\delta(t - z)$$

$$E_y = qy\left[\frac{1}{\gamma^2 R_v^3}\right] \xrightarrow[v \to 1]{} qy\left[\frac{2\delta(t - z)}{r_\perp^2}\right]$$

$$E_z = q(z - vt)\frac{1}{\gamma^3 R_v^3} \xrightarrow[v \to 1]{} \emptyset$$

$$B_x = -vE_y$$

$$B_y = vE_x$$

$$B_z = \emptyset$$

b) $$4\pi J^0 = 4\pi\rho = \bar{\nabla}\cdot\bar{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \cancel{\frac{\partial E_z}{\partial z}}$$

$$= 2q\,\delta(z-t)\left\{\frac{\partial}{\partial x}\Big[\underbrace{\frac{x}{x^2+y^2}}_{\frac{x}{r_\perp}}\Big] + \frac{\partial}{\partial y}\Big[\underbrace{\frac{y}{x^2+y^2}}_{\frac{y}{r_\perp}}\Big]\right\}$$

write $\dfrac{x}{x^2+y^2} = \dfrac{1}{2}\dfrac{\partial}{\partial x}\left[\ln(x^2+y^2)\right]$

then

$$4\pi J^0 = 2q\,\delta(z-t)\left\{\underbrace{\overbrace{\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]}^{\text{"}\nabla^2_{2D}\text{"}} \ln(x^2+y^2)}_{2\pi\,\delta(x)\,\delta(y)}\right\}$$

$$\boxed{J^0 = q\,\delta(z-t)\,\delta(x)\,\delta(y)}$$ remember $c=1$

similarlly:

$$\boxed{J^\alpha = q\,V^\alpha\,\delta(x)\,\delta(y)\,\delta(z-t)}$$

c)

$$A_1^{\mu} = \begin{pmatrix} 2q\,\delta(ct-z)\ln\left(\lambda\sqrt{x^2+y^2}\right) \\ 0 \\ 0 \\ -2q\,\delta(ct-z)\ln\left(\lambda\sqrt{x^2+y^2}\right) \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \phi \\ -\bar{A}_1 \end{pmatrix}$$

$$E_x = -\left(\cancel{\partial^0 A^1} - \partial^1 A^0\right) = +2q\,\delta(ct-z)\frac{\partial}{\partial x}\ln\left(\lambda\sqrt{x^2+y^2}\right)$$

$$\boxed{\begin{array}{l} E_x = 2q\,\delta(ct-z)\cdot\dfrac{x}{x^2+y^2} \\ \text{by symmetry} \\ E_y = 2q\,\delta(ct-z)\dfrac{y}{x^2+y^2} \end{array}}$$

works!

$$E_z = -\left(\partial^0 A^3 - \partial^3 A^0\right) A^3$$ but in this case $\partial^0 A^3 = +\partial^3 A^0$

so $\boxed{E_z = \varnothing}$

$$\boxed{\begin{array}{l} B_x = \partial^3 A^2 - \partial^2 A^3 = 2q\,\delta(ct-z)\dfrac{-y}{x^2+y^2} \\ B_y = 2q\,\delta(ct-z)\dfrac{x}{x^2+y^2} \\ B_z = \varnothing \end{array}}$$

works!

$\bar{A}_1 - \bar{A}_2 = \bar{\nabla}\phi$ if $\phi = -2q\,\Theta(ct-z)\ln(r_\perp\lambda) \Rightarrow \bar{A}_2 = -2\dfrac{\Theta(ct-z)}{x^2+y^2}(x\hat{x}+y\hat{y})$

4) $\begin{cases} \bar E' = \gamma(\bar E + \bar v \times \bar B) \\ \bar B' = \gamma(\bar B - \bar v \times \bar E) \end{cases}$

Given $\bar E$ and $\bar B$ we want some $\bar v$ such $\bar E' \times \bar B'$

Since we know $\bar v$ will be a <u>vector</u> then it only can point in $\bar E \times \bar B$ direction! because $\bar E$ is a <u>vector</u> and $\bar B$ is a <u>pseudo</u>vector.

so $\bar v = f \cdot (\bar E \times \bar B)$

$$\bar 0 \underset{\substack{\uparrow \\ \text{want}}}{\equiv} \bar E' \times \bar B' = \gamma^2 \Big[\big(\bar E + f \underbrace{(\bar E \times \bar B) \times \bar B}_{(\bar B\cdot\bar E)\bar B - (\bar B\cdot\bar B)\bar E}\big) \times \big(\bar B - f \underbrace{(\bar E \times \bar B) \times \bar E}_{(\bar E\cdot\bar E)\bar B - (\bar E\cdot\bar B)\cdot\bar E}\big)$$

$A \times (B \times C) = \bar B(A\cdot C) - \bar C(A\cdot B)$

$$\bar 0 = \cancel{\gamma^2}\Big[\big((1-fB^2)\bar E + f(\bar B\cdot\bar E)\bar B\big) \times \big((1-fE^2)\bar B + f(\bar E\cdot\bar B)\bar E\big)\Big]$$

$$\bar 0 = (1-fB^2)(1-fE^2)\,\bar E \times \bar B + \cancel{(1-fB^2)f(\bar E\cdot\bar B)\bar E\times\bar E} + f^2(E\cdot B)^2\,\bar B \times \bar E \quad \cancel{(1-fE^2)f(\bar B\cdot\bar E)\bar B\times\bar B}$$

$$\bar 0 = \underbrace{\Big[(1-fB^2)(1-fE^2) - f^2(\bar E\cdot\bar B)^2\Big]}_{=0}\cdot \bar E \times \bar B$$

$$1 - fE^2 - fB^2 + f^2B^2E^2 - f^2(\bar E\cdot\bar B)^2 = 0$$

$$f^2\big(B^2E^2 - (\bar E\cdot\bar B)^2\big) - f(E^2+B^2) + 1 = 0$$

solve for $f = \dfrac{E^2+B^2 \pm \sqrt{(E^2+B^2)^2 - 4\left[B^2E^2 - (\bar E\cdot\bar B)^2\right]}}{2(B^2E^2 - (\bar E\cdot\bar B)^2)}$ (*)

f exist if (*) > 0

indeed! (*) $= (E^2-B^2)^2 + 4(\bar E\cdot\bar B)^2$ is always positive

inside the root: $(E^2-B^2)^2 + 4(\bar E\cdot\bar B)^2 \geq 0$

if $= 0$ then $|\bar v| = c$ no boost possible

5) In any system $E^2 - p^2c^2$ is the same

Initial condition: two photons of energy

$E_{1\gamma} = 2.5\times10^4$ eV $\gamma_1 \longrightarrow \quad \longleftarrow \gamma_2$

$$P_i = \begin{pmatrix} E_{1\gamma}/c \\ E_{1\gamma}/c \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E_{2\gamma}/c \\ -E_{2\gamma}/c \\ 0 \\ 0 \end{pmatrix} \quad E_{2\gamma} \text{ (unknown)}$$

$$\vec{P}_{1\gamma} = E_{1\gamma}/c\ \hat{x}$$

$$\vec{P}_{2\gamma} = -E_{2\gamma}/c\ \hat{x}$$

After collision:

~~two rest~~ electron-positron pair at rest (relative)

with masses $mc^2 = 0.511$ MeV

$$P_f' = \begin{pmatrix} 2mc^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ at some frame $P_{e^-} = P_{e^+} = 0$ and $E_{total} = 2mc^2$

$$P_i = P_f \quad \text{and} \quad \| P_f \| = \| P_f' \|$$

$$\text{so} \quad \| P_i \| = \| P_f' \|$$

$$\downarrow$$

$$\underbrace{(E_{1\gamma} + E_{2\gamma})^2 - (E_{1\gamma} - E_{2\gamma})^2}_{4E_{1\gamma}E_{2\gamma}} = \underbrace{(2mc^2)^2}_{4(mc^2)^2}$$

$$E_{2\gamma} = (mc^2)^2/E_{1\gamma} \Longleftarrow$$

a) $E_{1\gamma} = 2.5\times10^{-4}$ eV $\rightarrow$ $E_{2\gamma} = 1.04\times10^{15}$ eV

b) $E_{1\gamma} = 500$ eV $\rightarrow$ $E_{2\gamma} = 5.22\times10^{8}$ eV