PS#6

1)

$$\begin{cases} \bar{x}' = \bar{x} + (\gamma_0 - 1) \dfrac{\bar{x}\cdot\bar{v}_0}{v_0^2}\,\bar{v}_0 - \gamma_0 \bar{v}_0 t \\ t' = \gamma_0 \left(t - \dfrac{\bar{v}_0 \cdot \bar{x}}{c^2} \right) \end{cases}$$

$$\bar{v}' \underset{\text{def}}{\equiv} \frac{d\bar{x}'}{dt'}$$

$$d\bar{x}' = d\bar{x} + (\gamma_0 - 1)\, d\bar{x}\cdot\frac{\bar{v}_0}{v_0^2}\,\bar{v}_0 - \gamma_0 \bar{v}_0\, dt$$

$$dt' = \gamma_0 \left(dt - \frac{\bar{v}_0 \cdot d\bar{x}}{c^2} \right)$$

$$\frac{d\bar{x}'}{dt'} = \frac{d\bar{x} + (\gamma_0 - 1)\, d\bar{x}\cdot\frac{\bar{v}_0}{v_0^2}\,\bar{v}_0 - \gamma_0 \bar{v}_0\, dt}{\gamma_0 \left(dt - \frac{\bar{v}_0 \cdot d\bar{x}}{c^2} \right)}$$

$\downarrow$ ~~mu~~ multipling by $\dfrac{dt}{dt} = 1$

$$\frac{d\bar{x}'}{dt'} = \frac{\frac{d\bar{x}}{dt} + (\gamma_0 - 1)\, \frac{d\bar{x}}{dt}\cdot\frac{\bar{v}_0}{v_0^2}\,\bar{v}_0 - \gamma_0 \bar{v}_0}{\gamma_0 \left(1 - \frac{\bar{v}_0 \cdot \frac{d\bar{x}}{dt}}{c^2} \right)}$$

$$\boxed{\bar{v}' = \frac{\vec{v} + (\gamma_0 - 1)\,\frac{\bar{v}\cdot\bar{v}_0}{v_0^2}\,\bar{v}_0 - \gamma_0 \bar{v}_0}{\gamma_0 \left(1 - \bar{v}_0 \cdot \bar{v}/c^2 \right)}} \Rightarrow$$

check; if $\bar{v} \parallel \bar{v}_0$

$$\bar{v}' = \frac{\bar{v} - \bar{v}_0}{1 - \frac{\bar{v}_0 \bar{v}_\parallel}{c^2}}$$

known formula!

$$\underset{\text{def}}{\bar{a}' \equiv} \frac{d\bar{\mathbf{v}}'}{dt'}$$

$$d\bar{v}' = \frac{d\bar{v} + (\gamma_0 - 1)\dfrac{d\bar{v}\cdot\bar{v}_0}{v_0^2}\,\bar{v}_0}{\gamma_0\left(1 - \dfrac{\bar{v}_0\cdot\bar{v}}{c^2}\right)} + \frac{\bar{v} + (\gamma_0 - 1)\dfrac{\bar{v}\cdot\bar{v}_0}{v_0^2}\bar{v}_0 - \gamma_0\bar{v}_0}{+\left[\gamma_0\left(1 - \dfrac{\bar{v}_0\cdot\bar{v}}{c^2}\right)\right]^2}\,\gamma_0\frac{\bar{v}_0\cdot d\bar{v}}{c^2}$$

while

$$dt' = \gamma_0\left(dt - \frac{\bar{v}_0\cdot d\bar{x}}{c^2}\right)$$

$$\frac{d\bar{v}'}{dt'} = \frac{d\bar{v} + (\gamma_0 - 1)\dfrac{d\bar{v}\cdot\bar{v}_0}{v_0^2}\,\bar{v}_0}{\gamma_0^2\left(1 - \dfrac{\bar{v}_0\cdot\bar{v}}{c^2}\right)\left(dt - \dfrac{\bar{v}_0\cdot d\bar{x}}{c^2}\right)} + \frac{\bar{v} + (\gamma_0 - 1)\dfrac{\bar{v}\cdot\bar{v}_0}{v_0^2}\bar{v}_0 - \gamma_0\bar{v}_0}{\gamma_0^3\left(1 - \dfrac{\bar{v}_0\cdot\bar{v}}{c^2}\right)^2\left(1 - \dfrac{\bar{v}_0\cdot d\bar{x}}{c^2}\right)}\,\gamma_0\frac{\bar{v}_0\cdot d\bar{v}}{c^2}$$

multipling by $\dfrac{dt}{dt} = 1$ (same trick as before)

$$\boxed{\frac{d\bar{v}'}{dt'} = \bar{a}' = \frac{\bar{a} + (\gamma_0 - 1)\dfrac{\bar{a}\cdot\bar{v}_0}{v_0^2}\,\bar{v}_0}{\gamma_0^2\left(1 - \dfrac{\bar{v}_0\cdot\bar{v}}{c^2}\right)^2} + \frac{\bar{v} + (\gamma_0 - 1)\dfrac{\bar{v}\cdot\bar{v}_0}{v_0^2}\bar{v}_0 - \gamma_0\bar{v}_0}{\gamma_0^3\left(1 - \dfrac{\bar{v}_0\cdot\bar{v}}{c^2}\right)^3}\,\frac{\gamma_0\bar{v}_0\cdot\bar{a}}{c^2}}$$

~~if $\bar{a} \parallel \bar{v} \parallel \bar{v}_0$~~

$$\cancel{\bar{a}' = \frac{\bar{a} + (\gamma_0 - 1)\bar{a}}{\gamma_0^2\left(1 - \frac{\bar{v}_0\cdot\bar{v}}{c^2}\right)^2} + \frac{\bar{v} + (\gamma_0 - 1)\bar{v}_0 - \gamma_0\bar{v}_0}{\gamma_0^3\left(1 - \frac{\bar{v}_0\cdot\bar{v}}{c^2}\right)^3}\frac{\gamma_0 v_0 a}{c^2} = \frac{\bar{a}}{\gamma^3}}$$

2) a)

$$X = (t;\ \bar{x}) \xrightarrow{\text{rotation}} (t;\ \bar{R}(\bar{x})) = X'$$

$$X^{\mu} X_{\mu} = t^2 - \bar{x}\cdot\bar{x} = t^2 - |\bar{x}|^2$$

$$X'^{\mu} X'_{\mu} = t^2 - \bar{R}(\bar{x})\cdot\bar{R}(\bar{x}) = t^2 - |\bar{R}(\bar{x})|^2$$

but $|\bar{R}(\bar{x})|^2 = |\bar{x}|^2$ by def. of rotation

so

$$X'^{\mu} X'_{\mu} = t^2 - |\bar{x}|^2 = X^{\mu} X_{\mu}$$

then rotation is a Lorentz tranformation.

~~L is a Lorentz transformation only if~~ $\cancel{g_{\mu\nu} = L_{\mu}{}^{\alpha} g_{\alpha\beta} L^{\beta}{}_{\nu}}$

~~in particular~~

$$\cancel{g_{00} = L_0{}^{\alpha} g_{\alpha\beta} L^{\beta}{}_0 = L_0{}^0 L^0{}_0 - L_0{}^1 L^1{}_0 - L_0{}^2 L^2{}_0 - L_0{}^3 L^3{}_0}$$

2) L is a Lorentz transformation only if

$$\mathbb{g} = \mathbb{L}^t \, \mathbb{g} \, \mathbb{L}$$

$$g_{\mu\nu} = (L^t)_\mu{}^\alpha \, g_{\alpha\beta} \, L^\beta{}_\nu$$

$$= L_\alpha{}^\mu \, g_{\alpha\beta} \, L^\beta{}_\nu$$

in particular

$$g_{00} = L_\alpha{}^0 g_{\alpha\beta} L^\beta{}_0 = L_0{}^0 g_{00} L^0{}_0 + L_1{}^0 g_{11} L^1{}_0 + \cdots$$

$$\downarrow$$

$$1 = L_0{}^0 L^0{}_0 - L_1{}^0 L^1{}_0 - \cdots$$

$$1 = (L^0{}_0)^2 - (L^1{}_0)^2 - (L^2{}_0)^2 - (L^3{}_0)^2$$

so $$\boxed{(L^0{}_0)^2 \geq 1}$$

3) From problem 1) take $c = 1$

$$\boxed{v = \frac{v_1 + v_2}{1 + v_1 v_2}}$$

with constrain $v_1 < 1$, $v_2 < 1$

↓ change notation

$$z = \frac{x+y}{1+xy}$$

with constrains $|x| < 1$, $|y| < 1$

find extrema:

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{1}{1+xy} - \dfrac{x+y}{(1+xy)^2}\, y = 0 \\[2ex] \dfrac{\partial z}{\partial y} = \dfrac{1}{1+xy} - \dfrac{x+y}{(1+xy)^2}\, x = 0 \end{cases}$$

$$\begin{cases} 1 = \dfrac{x+y}{1+xy}\, y \\[2ex] 1 = \dfrac{x+y}{1+xy}\, x \end{cases} \Rightarrow \begin{cases} x = y \\[2ex] 1 = \dfrac{2x}{1+x^2}\, x \end{cases} \Rightarrow \begin{cases} x = 1 \wedge y = 1 \\ \vee \\ x = -1 \wedge y = -1 \end{cases}$$

↓

there are no extrema inside the box $(-1; 1) \times (-1; 1)$

in the boundaries: $|z| \equiv 1$

$\hookrightarrow$ $|z| < 1$ if $-1 < x < 1$ and $-1 < y < 1$

Let $[\mathbb{g}]_{ij} = g_{ij}$

$[\tilde{\mathbb{g}}]_{ij} = g^{ij}$

$[\mathbb{1}]_{ij} = \delta_i{}^j$

$$\mathbb{g}\,\tilde{\mathbb{g}} = \mathbb{1}$$

$$\mathbb{g} = \mathbb{L}^t\,\mathbb{g}'\,\mathbb{L}$$ ← transform like a covariant tensor

$$\mathbb{L}^t\,\mathbb{g}'\,\mathbb{L}\,\tilde{\mathbb{g}} = \mathbb{1}$$

$$\underbrace{\cancel{\mathbb{L}\,\mathbb{L}^t}}_{\mathbb{1}}\,\mathbb{g}'\,\mathbb{L}\,\tilde{\mathbb{g}}\,\mathbb{L}^t = \underbrace{\cancel{\mathbb{L}\,\mathbb{1}\,\mathbb{L}^t}}_{\mathbb{1}}$$

$$\mathbb{g}'\,\mathbb{L}\,\tilde{\mathbb{g}}\,\mathbb{L}^t = \mathbb{1}$$

so $\tilde{\mathbb{g}}' = \mathbb{L}\,\mathbb{g}\,\mathbb{L}^t$ ← transforms like a contravariant tensor.

where $\tilde{\mathbb{g}}\,\tilde{\mathbb{g}}' = \mathbb{1}$

4) b)

$$T_i{}^j = \partial_i A^j = \frac{\partial}{\partial x^i} A^j$$

$$T'_i{}^j = \frac{\partial}{\partial x'^i} A'^j$$

$$\frac{\partial}{\partial x'^i} = \frac{\partial}{\partial x^k} \frac{\partial x^k}{\partial x'^i}$$

and

$$A'^j = \frac{\partial x^j}{\partial x'^i} A^i$$

$$T'_i{}^j = \frac{\partial}{\partial x^k} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^j}{\partial x'^i} A^i$$

in $\frac{\partial x^k}{\partial x'^i}$ is constant then $\frac{\partial}{\partial x^k}\frac{\partial x^k}{\partial x'^i} = \frac{\partial x^k}{\partial x'^i}\frac{\partial}{\partial x^k}$

then

$$T'_i{}^j = \frac{\partial x^k}{\partial x'^i}\left[\frac{\partial}{\partial x^k} A^i\right]\frac{\partial x^j}{\partial x'^i} = \frac{\partial x^k}{\partial x'^i}\, T_k{}^i\, \frac{\partial x^j}{\partial x'^i} = L^k{}_i\, T_k{}^i\, L^j{}_i$$

transforms like a mixed tensor! ✓

but if Jacobian is not constant ⇒ there are two aditional terms that come from appling the chain rule.

c) $\mathbb{g}$ metric is a symmetric and real matrix
so there exist $\mathbb{d}$ and $\mathbb{b}$ such that:

$$\mathbb{d} = \mathbb{b}\,\mathbb{g}\,\mathbb{b}^t$$

where $\mathbb{d}$ is diagonal and $\mathbb{b}^t\mathbb{b} = \mathbb{1}$

$$\mathbb{g} = \mathbb{b}^t\,\mathbb{d}\,\mathbb{b}$$

if $\mathbb{g}$ is definite positive then the diagonal element of $\mathbb{d}$ are positive
and we can define a unique $\sqrt{\mathbb{d}}$ also diagonal and definite positive such that $\sqrt{\mathbb{d}}\sqrt{\mathbb{d}} = \mathbb{d}$
and $(\sqrt{\mathbb{d}})^t = \sqrt{\mathbb{d}}$ because it is diagonal

$$\mathbb{g} = \mathbb{b}^t\sqrt{\mathbb{d}}\;\mathbb{1}\;\sqrt{\mathbb{d}}\,\mathbb{b}$$

$$\mathbb{g} = (\sqrt{\mathbb{d}}\,\mathbb{b})^t\;\mathbb{1}\;(\sqrt{\mathbb{d}}\,\mathbb{b})$$

$$\mathbb{x}^t\mathbb{g}\,\mathbb{x} = \underbrace{(\sqrt{\mathbb{d}}\,\mathbb{b}\,\mathbb{x})^t}_{\mathbb{x}'^t}\;\mathbb{1}\;\underbrace{(\sqrt{\mathbb{d}}\,\mathbb{b}\,\mathbb{x})}_{\mathbb{x}'}$$

so in the coordinated system $\mathbb{x}' = (\sqrt{\mathbb{d}}\,\mathbb{b})\,\mathbb{x}$
the metric is the identity

→

c) cont'd

nor definite negative.

if $\mathbb{g}$ is not definite positive $\mathbb{d}$ has the general form

$$\mathbb{d} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \qquad \text{with } \lambda_1 > 0, \ \lambda_2 > 0$$

instead of defining $\sqrt{\mathbb{d}}$ lets define "$\sqrt{|\mathbb{d}|}$"

as $\sqrt{|\mathbb{d}|} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}$ and $\mathbb{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

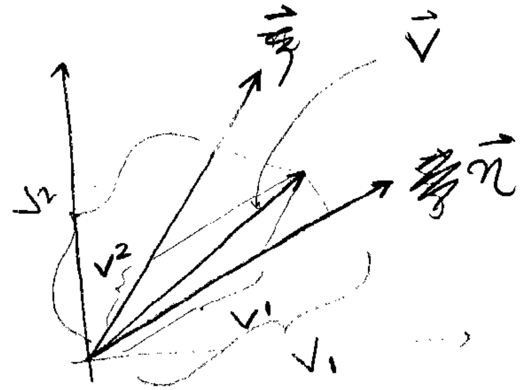
note that $\left(\sqrt{|\mathbb{d}|}\right)^t = \sqrt{|\mathbb{d}|}$

so $\mathbb{g} = \left(\sqrt{|\mathbb{d}|}\,\mathbb{b}\right)^t \mathbb{J} \left(\sqrt{|\mathbb{d}|}\,\mathbb{b}\right)$

so in the coordinate system $\mathbb{x}' = \left(\sqrt{|\mathbb{d}|}\,\mathbb{b}\right)\mathbb{x}$

the metric is $\mathbb{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

d)



$$\vec V = \underset{\uparrow}{V^1}\,\hat\eta + \underset{\uparrow}{V^2}\,\hat\xi$$

contravariant components by definition

we also want some axes $\hat\gamma$ and $\hat\mu$

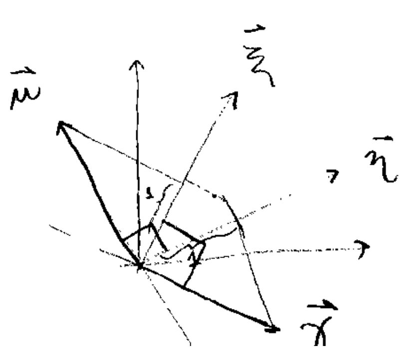
where $\vec V = V_1\vec\gamma + V_2\vec\mu$

and we will find $\vec\gamma$ and $\vec\mu$ by asking the desired expression for $|\vec V| = V^1V_1 + V^2V_2$ for any vector $\vec V$

$$\vec V\cdot\vec V = V^1V_1\underbrace{(\hat\eta\cdot\vec\gamma)}_{=1} + \cancel{V^1V_2\underbrace{(\hat\eta\cdot\vec\mu)}_{=0}} + \cancel{V^2V_1\underbrace{(\hat\xi\cdot\vec\gamma)}_{=0}} + V^2V_2\underbrace{(\hat\xi\cdot\vec\mu)}_{=1}$$

$$\begin{pmatrix}\hat\eta\cdot\vec\gamma & \hat\eta\cdot\vec\mu\\ \hat\xi\cdot\vec\gamma & \hat\xi\cdot\vec\mu\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

this defines the vectors $\vec\gamma$ and $\vec\mu$ known as reciprocal vectors of $\hat\eta$ and $\hat\xi$



as $\vec\mu \perp \vec\eta$

V_1 is obtained projecting $\vec V$ onto $\vec\eta$ perpendicularly

and the same for V_2

so $\bar V = V_1\bar\gamma + V_2\bar\mu = V^1\hat\eta + V^2\hat\xi$

4) e)

$$A'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} A_{lk}$$

$$A'_{ij} = \pm \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} A_{kl}$$

symmetric case

$$\boxed{A'_{ij} = \pm A'_{ji}}$$

antisymmetric case

for mixed case

$$A'^{\ j}_{i} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} A_k^{\ l}$$

sym. case

$$= \pm \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} A_l^{\ k} \neq \pm A'^{\ j}_{i}$$

antisym. case

!

(anti)symmetry is
not well defined for mixed
(is not a covariant property;
it depends on the frame)

5)

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = \emptyset$$

$$\begin{cases} \partial_y E_z - \partial_z E_y + \partial_t B_x = \emptyset \\ \partial_z E_x - \partial_x E_z + \partial_t B_y = \emptyset \\ \partial_x E_y - \partial_y E_x + \partial_t B_z = \emptyset \end{cases}$$

$$\begin{cases} \partial_x = \gamma (\partial_x' - v \partial_t) \\ \partial_y = \partial_y' \\ \partial_z = \partial_z' \\ \partial_t = \gamma (\partial_t' - v \partial_x') \end{cases}$$

$$\begin{cases} \partial_x' (\gamma B_x) + \partial_y' (B_y) + \partial_z' (B_z) - \partial_t (v \gamma B_x) = \emptyset \\ \partial_y' (E_z) - \partial_z' (E_y) + \partial_t' (\gamma B_x) - \partial_x' (\gamma v B_x) = \emptyset \\ \partial_z' (E_x) - \partial_x' (\gamma E_z) + \partial_t' (v \gamma E_z) + \partial_t (\gamma B_y) - \partial_x' (\gamma v B_y) = \emptyset \\ \partial_x' (\gamma E_y) - \partial_t' (v \gamma E_y) - \partial_y' (E_x) + \partial_t' (\gamma B_z) - \partial_x' (v \gamma B_z) = \emptyset \end{cases}$$

choosing $E_x' = A E_x$ and $B_x' = B_x A$

$E_y' = \gamma (E_y - v B_z) A$ $\quad B_y' = \gamma (B_y' + v E_z) A$

$E_z' = \gamma (B_z - v E_y) A$ $\quad B_z' = \gamma (B_z' - v E_y) A$

(we can choose $A = 1$ for all systems)

we recover Maxwell equs for primed frame.