(J5.14) PS #5

1)

no free currents $\mapsto$ $\bar{\nabla} \times \bar{H} = \bar{0} \Rightarrow \exists \phi \,/\, \bar{H} = -\bar{\nabla}\Phi$

$\bar{\nabla} \cdot \bar{B} = 0$ and μ is constant in each region

$\Downarrow$

$$\bar{\nabla} \cdot \bar{H} = 0$$

$\longrightarrow$ $\nabla^2 \phi = 0$ in each region

Boundary conditions: $\bar{H}_{\parallel}$ and $\bar{B}_{\perp}$ continuous

are equivalent to

Φ ~~continuous~~ continuous

and $\mu^+ \frac{\partial \Phi}{\partial \rho}\Big|_{s^+} = \mu^- \frac{\partial \Phi}{\partial \rho}\Big|_{s^-}$

~~General solution~~

Boundary conditions at infinity

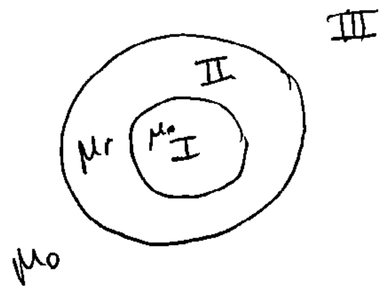
$\bar{H}(\rho \to \infty) = H_0 \hat{x} \Rightarrow \Phi(\rho \to \infty) = -H_0 x = -H_0 \rho \cos\phi$

General solution of Laplace equ. in each region:

$$\Phi = A + B \ln(\rho) + \sum_{n=1}^{\infty} (A_n \rho^n + B_n \rho^{-n}) \cos(n\phi) + \sum_{n=1}^{\infty} (C_n \rho^n + D_n \rho^{-n}) \sin(n\phi)$$

Because of boundary condition at infinity and the normal derivatives doesn't change the angular dependence the almost all the coefficients can be dropped:

$$\boxed{\begin{aligned} A_n &= 0 \quad \forall n \neq 1 \\ B_n &= 0 \quad \forall n \neq 1 \end{aligned}}$$



$$C_n = 0 \quad \forall n$$

$$D_n = 0 \quad \forall n$$

$$A = 0$$

$$B = 0$$

Moreover, in inner region $B_1^{I} = 0$

and in outer region $A_1^{III} = -H_0$

because of boundary condition.

So we have only 4 unknows:

A_1^{I}; B_1^{II}; A_1^{II}; B_1^{III}

↓

(field is uniform in the inner region)!

$\cancel{\Phi^{I} = B_n \rho^n \cos(n\phi)}$

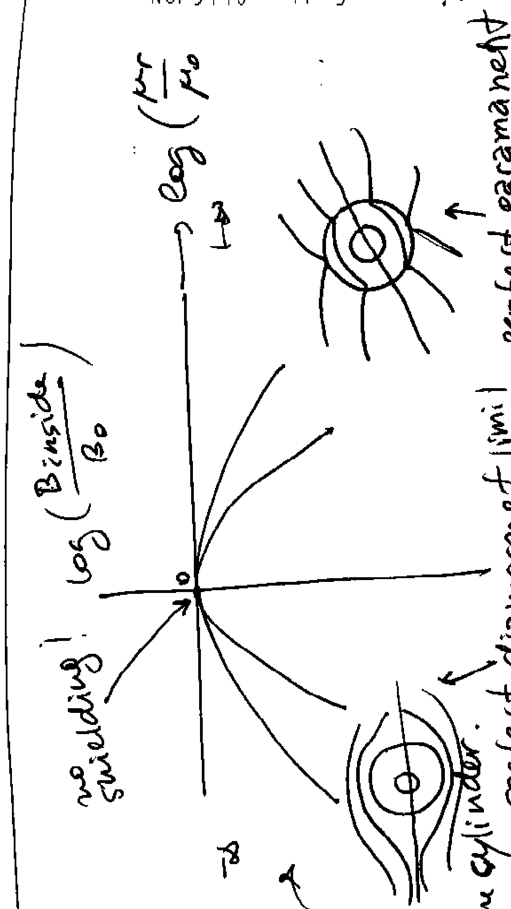
$$\Phi = \begin{cases} \left(A_1^{I}\rho + \cancel{B_1^{I}\rho^{-1}}\right)\cos\phi \\ \left(A_1^{II}\rho + B_1^{II}\rho^{-1}\right)\cos\phi \\ \left(\underbrace{A_1^{III}}_{-H_0}\rho + B_1^{III}\rho^{-1}\right)\cos\phi \end{cases}$$

$$\frac{\partial\Phi}{\partial\rho} \begin{cases} \left(A_1^{I} + \cancel{B}\right)\cos\phi \\ \left(A_1^{II} - B_1^{II}\rho^{-2}\right)\cos\phi \\ \left(\underbrace{A_1^{III}}_{H_0} - B_1^{III}\rho^{-2}\right)\cos\phi \end{cases}$$

matching with boundary conditions at $\rho = a$ & $\rho = b$

$$\begin{bmatrix} a & -a & -\frac{1}{a} & 0 \\ a & -\mu_r a & \frac{\mu_r}{a} & 0 \\ 0 & b & \frac{1}{b} & -\frac{1}{b} \\ 0 & \mu_r b & -\frac{\mu_r}{b} & \frac{1}{b} \end{bmatrix} \begin{bmatrix} A_1^{I} \\ A_1^{II} \\ B_1^{II} \\ B_1^{III} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{B_0}{\mu_0} b \\ -\frac{B_0}{\mu_0} b \end{bmatrix}$$

$(H_0 = \mu_0 B_0)$



Gives:

$$\boxed{A_1^{I} = \left(-\frac{B_0}{\mu_0}\right)\frac{4\mu_r^2 b^2}{b^2(\mu_r+1)^2 - a^2(\mu_r-1)^2}}$$

It is also the magnitude of $\bar{B}$ field inside the cylinder.

(J6.1)

2) a)

$$\psi(\bar{x},t) = \int \frac{[f(\bar{x}',t')]_{ret}}{|\bar{x}-\bar{x}'|} d^3\bar{x}'$$

for the (cylindrical) source $f(\bar{x}',t') = \delta(x')\delta(y')\delta(t')$

$$\psi(\bar{x},t) = \int \frac{\delta(x')\delta(y')\delta(t')}{|\bar{x}-\bar{x}'|} d^3x'$$

where $t = t' + \frac{|\bar{x}-\bar{x}'|}{c}$

$$= \int \frac{\delta(x')\delta(y')\delta\left(t - \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}/c\right)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} dx'dy'dt'$$

$$\psi(\bar{x},t) = \int_{-\infty}^{\infty} \frac{\delta\left(t - \sqrt{x^2+y^2+(z-z')^2}/c\right)}{\sqrt{x^2+y^2+(z-z')^2}} dz'$$

$(z' \to z + z')$

$$\psi\left(\begin{smallmatrix}x\\y\end{smallmatrix};t\right) = \int_{-\infty}^{\infty} \frac{\delta\left(t - \sqrt{x^2+y^2+z'^2}/c\right)}{\sqrt{x^2+y^2+z'^2}} dz'$$

$$\psi(\rho;t) = \int_{-\infty}^{\infty} \frac{\delta\left(t - \sqrt{\rho^2+z'^2}/c\right)}{\sqrt{\rho^2+z'^2}} dz'$$

in general

$$\delta(f(z)) = \sum_i \frac{\delta(z - z_i)}{|f'(z_i)|} \quad \text{where } f(z_i) = 0$$

$$f(z') = t - \frac{\sqrt{\rho^2 + z'^2}}{c} \qquad f'(z') = -\frac{1}{\cancel{2}}(\rho^2 + z'^2)^{-1/2}\frac{\cancel{2}z'}{c}$$

$$t - \frac{\sqrt{\rho^2 + z'^2}}{c} = 0$$

$$c^2t^2 = \rho^2 + z'^2$$

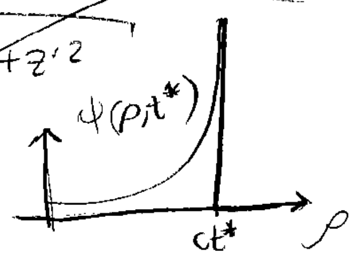
$$c^2t^2 - \rho^2 = z'^2$$

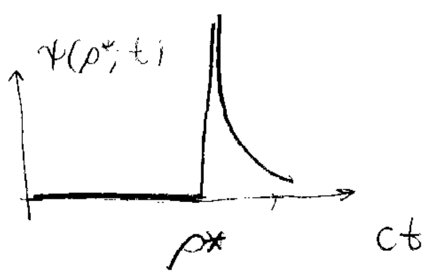
$$z_1 = +\sqrt{c^2t^2 - \rho^2}$$

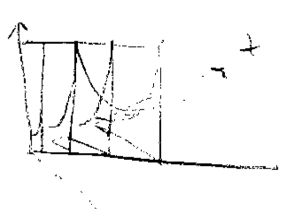
$$z_2 = -\sqrt{c^2t^2 - \rho^2}$$

$$\psi(\rho;t) = \int_{-\infty}^{\infty} \frac{\delta\left(z' - \sqrt{c^2t^2 - \rho^2}\right) + \delta\left(z' + \sqrt{c^2t^2 - \rho^2}\right)}{\left|-\frac{1}{c} z' \cancel{(\rho^2 + z'^2)^{-1/2}}\right| \cancel{\sqrt{\rho^2 + z'^2}}}\, dz'$$

$$\boxed{\psi(\rho,t) = \frac{2e}{\sqrt{c^2t^2 - \rho^2}}\,\Theta(c^2t^2 - \rho^2)}$$







b) "sheet" source $f(\bar{x},t) = \delta(z)\delta(t)$

$$\psi(\bar{x};t) = \int \frac{\delta(z')\,\delta(t')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\,d^3r$$

$\cancel{\psi(z,t) = \int \delta(z')\delta(t')}$

where $t = t' + \sqrt{(x-x')^2+(y'-y')^2+(z-z')^2}/c$

$$\psi(\bar{x},t) = \int \frac{\delta(z')\,\delta\left(t-\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}/c\right)}{\sqrt{}}\,dx'dy'dz'$$

shifting the origin:

$$\psi(\bar{x},t) = \int \frac{\delta(z')\,\delta\left(t-\sqrt{x'^2+y'^2+(z-z')^2}/c\right)}{\sqrt{x'^2+y'^2+(z-z')^2}}\,dx'dy'dz'$$

$$\psi(z,t) = \iint \frac{\delta\left(t-\sqrt{x'^2+y'^2+z^{2'}}/c\right)}{\sqrt{x'^2+y'^2+z^2}}\,dx'dy'$$

$$= \int \frac{\delta\left(t-\sqrt{\rho'^2+z^{2'}}/c\right)}{\sqrt{\rho'^2+z^2}}\,2\pi\rho'd\rho'$$

$$\delta(f(z)) = \sum_i \frac{\delta(z - z_i)}{|f'(z_i)|}$$

$$\delta\left(t - \sqrt{\rho'^2 + z^2}/c\right) =$$

$$\frac{\delta\left(\rho' - \sqrt{c^2t^2 - z^2}\right) + \delta\left(\rho' + \sqrt{c^2t^2 - z^2}\right)}{2\,(\rho'^2 + z^2)^{-1/2}\,\rho'/c}$$

$$\psi(z,t) = \int \frac{\delta(\rho' - \sqrt{t^2 - z^2})}{\cancel{2(\rho'^2 + z^2)^{-1/2}}\,\frac{\cancel{\rho'}}{c}\,\cancel{(\rho'^2 + z^2)^{1/2}}}\,2\pi\cancel{\rho'}\,d\rho' + \int \frac{\delta(\rho' + \sqrt{t^2 - z^2})}{\cancel{2(\rho'^2 + z^2)^{-1/2}}\,\frac{\cancel{\rho'}}{c}\,\cancel{(\rho'^2 + z^2)^{1/2}}}\,2\pi\cancel{\rho'}\,d\rho'$$

$$\boxed{\psi(z,t) = \frac{4\pi c}{2}\,\Theta\left(c^2t^2 - z^2\right)}$$

3) a)

Using one of the possible definitions of $\bar{E}_{//}$

$$\bar{E}_{//}(\bar{r}) = -\frac{1}{4\pi}\vec{\nabla}\int \frac{\vec{\nabla}'\cdot\vec{E}(\bar{r}')}{|\bar{r}-\bar{r}'|}\,d\bar{r}^3$$

We don't know $\bar{E}$ yet but thanks to the first ~~...~~ Maxwell equation we know $\bar{\nabla}\cdot\bar{E} = 4\pi\rho$

so

$$\bar{E}_{//}(\bar{r},t) = -\cancel{\frac{1}{4\pi}}\vec{\nabla}\int \frac{\cancel{4\pi}\,\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}\,d\bar{r}^3$$

(There is no need to work in any particular Gauge.)

$\rho(\bar{r}') = q\,\delta\left(\bar{r}' - \bar{x}(t)\right)$ where $\bar{x}(t)$ is the possition of particle.

integrating

$$\bar{E}_{//}(\bar{r},t) = -\vec{\nabla}\left[\frac{q}{|\bar{r}-\bar{x}(t)|}\right]$$

to first order in $\frac{\bar{x}(t)}{r}$:

$$\frac{1}{|\bar{r}-\bar{x}(t)|} = \frac{1}{r} + \frac{\bar{r}\cdot\bar{x}(t)}{r^3} + \cdots$$

$$\vec{E}_{//}(\bar{r},t) = -q\,\bar{\nabla}\left[\frac{1}{r} + \frac{\bar{r}\cdot\bar{x}(t)}{r^3} + \cdots\right]$$

the first term is independent of time so it is probably what Littlejohn calls $E_{\parallel}(x;0)$

$$\vec{E}_{\parallel}(\bar{r};t) = \underbrace{-\bar{\nabla}\left(\frac{q}{r}\right)}_{\vec{E}_{\parallel}(\bar{r};0)} + \underbrace{-\bar{\nabla}\left(\frac{\vec{r}\cdot\bar{x}(t)}{r^3}\right)q}_{\delta\bar{E}_{\parallel}(\bar{r};t)}$$

$$\delta\bar{E}_{\parallel}(\bar{r};t) = +\bar{\nabla}\left(\bar{\nabla}\frac{1}{r}\cdot\bar{x}(t)\right)q$$

$$\left(\delta\bar{E}_{\parallel}(\bar{r};t)\right)_i = +\partial_i\left(\partial_j\frac{1}{r}\right)x_j(t)\,q$$

$$= +\left(\partial_i\partial_j\frac{1}{r}\right)x_j(t)\,q$$

$$\delta\bar{E}_{\parallel}(\bar{r};t) = +\underbrace{\left(\bar{\nabla}\otimes\bar{\nabla}\frac{1}{r}\right)}_{=4\pi\Delta^{\perp}(\bar{r})\ (\text{if } r\neq 0)}:\bar{x}(t)\,q$$

$$\boxed{\delta\bar{E}_{\parallel} = 4\pi q\,\Delta^{\perp}(\bar{r}) : \bar{x}(t)}$$

← This is an instantaneous field so it must be canceled by other instantaneous contribution of $\bar{E}_{\perp}$

b)

$$\bar{J}(\bar{r},t) = q\,\dot{\bar{x}}(t)\,\delta(\bar{r}-\bar{x}(t))$$

$$\bar{J}_\perp(\bar{r},t) = \int \Delta^\perp(\bar{r}-\bar{r}') : \bar{J}(\bar{r}')\, d^3r'$$

$$= \int \Delta^\perp(\bar{r}-\bar{r}') : \delta(\bar{r}'-\bar{x}(t))\; q\,\dot{\bar{x}}(t)\, d^3r'$$

$$\boxed{\bar{J}_\perp(\bar{r},t) = q\, \Delta^\perp(\bar{r}-\bar{x}(t)) : \dot{\bar{x}}(t)}$$

the lowest order in the displacement is the linear one; this mean that in this order we cantake only the zero^th order of $\Delta^\perp$

$$\boxed{\bar{J}_\perp(\bar{r},t) = q\,\Delta^\perp(\bar{r}) : \dot{\bar{x}}(t)}$$

$$\frac{1}{c}\frac{\partial \bar{E}}{\partial t} = \bar{\nabla}\times\bar{B} + 4\pi\frac{\bar{J}}{c}$$

taking the perpendicular component

$$\cancel{\frac{1}{c}}\frac{\partial \bar{E}_\perp}{\partial t} = \underbrace{\cancel{(\bar{\nabla}\times\bar{B})_\perp}}_{\bar{\nabla}\times\bar{B} \text{ is already perpendicular}} + \frac{4\pi}{\cancel{c}}\,\bar{J}_\perp$$

sup. it is zero (for a moment see point c))

$$\frac{\partial \bar{E}_\perp}{\partial t} = -4\pi \bar{J}_\perp$$

$$\frac{\partial \bar{E}_\perp}{\partial t} = -4\pi q \, \Delta^\perp(\bar{r}) : \dot{\bar{x}}(t)$$

we know that the initial condition is $\bar{E}_\perp(t=0) = \bar{0}$

so ; integrating :

$$\boxed{\bar{E}_\perp(\bar{r};t) = -4\pi q \, \Delta^\perp(\bar{r}) : \bar{x}(t)}$$

which exactly cancels $\delta\bar{E}_\parallel$

so there is no instantaneous field

c) $$\frac{1}{c}\frac{\partial}{\partial t}\bar{E}_\perp = \bar{\nabla}\times\bar{B} - 4\pi\frac{\bar{J}_\perp}{c}$$

taking curl $$\underbrace{\bar{\nabla}\times\bar{\nabla}\times\bar{B}}_{-\nabla^2\bar{B} + \bar{\nabla}(\cancel{\bar{\nabla}\cdot\bar{B}}) \,\to\, 0} = \frac{1}{c}\frac{\partial}{\partial t}\bar{\nabla}\times\bar{E}_\perp + 4\pi\frac{\bar{\nabla}}{c}\times\bar{J}_\perp$$

and

$$(\bar{\nabla}\times\bar{E}) = \bar{\nabla}\times\bar{E}_\perp = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t}$$

so

$$\underbrace{-\nabla^2\bar{B} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{B}}_{\text{wave equation}} + 4\pi\underbrace{\bar{\nabla}\times\bar{J}_\perp}_{\bar{\nabla}\times\bar{J}}$$

~~≠0~~ only

wave equation

finite propagation velocity → $\bar{B}(\bar{r}) = \emptyset$ far from origin

only ≠∅ near the particle

5) a)

event 1 at time t_1 position $\bar{x}_1$

event 2 at time t_2 position $\bar{x}_2$ and $t_1 = t_2 = t$

Let's consider a ~~the~~ Lorentz tranformation to a frame moving a velocity $\bar{v}$:

event 1 at time $t_1' = \gamma\left(t - \frac{\bar{v}\cdot\bar{x}_1}{c^2}\right)$

event 2 at time $t_2' = \gamma\left(t - \frac{\bar{v}\cdot\bar{x}_2}{c^2}\right)$

$$t_2' - t_1' = -\frac{\gamma}{c^2}\,\bar{v}\cdot\left(\bar{x}_2 - \bar{x}_1\right)$$

$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ is not bounded for a velocity v near c.

$\Downarrow$

$\boxed{t_2' - t_1' \text{ has no limit}}$

b) The time-space distance $\Delta s^2 = \Delta t^2 c^2 - \Delta x^2$ is a Lorentz invariant

in fact $\Delta s^2 = -(\bar{x}_1 - \bar{x}_2)^2$ so

$$\Delta s^2 = \Delta s'^2 = \Delta t'^2 c^2 - \Delta x'^2$$

$$-(\bar{x}_1 - \bar{x}_2)^2 = \Delta t'^2 c^2 - \Delta x'^2$$

we showed that is goes from 0 to infinity

so $\Delta x'^2$ <u>varies between $(\bar{x}_1 - \bar{x}_2)^2$ to infinity</u> QED.