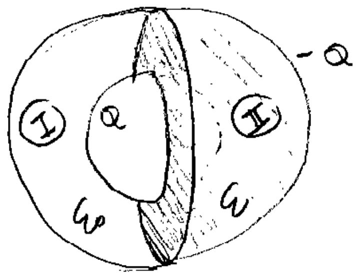


(J4.10)

1) a)

$\bar{D}$ field only "see" free charges; if we sup that is radial an ~~shape~~ spherically symmetric in each region:

$$\bar{D}_I = D_I(r)\,\hat{r}$$

$$\bar{D}_{II} = D_{II}(r)\,\hat{r}$$

$$\bar{\nabla}\cdot\bar{D} = \rho_{free} \Rightarrow \left[D_I(r) + D_{II}(r)\right] 2\pi r^2 = Q$$

$$\boxed{D_I(r) + D_{II}(r) = \frac{Q}{2\pi r^2}}$$

in I $\quad \bar{E}_I = \dfrac{\bar{D}_I}{\varepsilon_0}$ $\qquad$ in II $\quad \bar{E}_{II} = \dfrac{\bar{D}_{II}}{\varepsilon}$

boundary conditions

$$\bar{E}_{I//}\Big|_{interfase} = \bar{E}_{II//}\Big|_{interfase}$$

but $\bar{E}_{//} = \bar{E}$ (because is radial)

$$\boxed{\frac{D_I}{\varepsilon_0} = \frac{D_{II}}{\varepsilon}} \longrightarrow D_I(r) + \frac{\varepsilon}{\varepsilon_0} D_I(r) = \frac{Q}{2\pi r^2}$$

$$\boxed{D_I(r) = \frac{\varepsilon_0 Q}{(\varepsilon_0 + \varepsilon)\, 2\pi r^2}} \longrightarrow$$

$$\boxed{D_{II}(r) = \frac{\varepsilon Q}{(\varepsilon_0 + \varepsilon)\, 2\pi r^2}}$$

in any case $$\boxed{E_I(r) = E_{II}(r) = \frac{Q}{(\varepsilon_0 + \varepsilon)\, 2\pi r^2}}$$

b) The free charge is $\sigma_{free} = \bar{D} \cdot \hat{n} \Big|_{r=a}$

$$\boxed{\sigma_I^{free} = \frac{\varepsilon_0 Q}{(\varepsilon_0 + \varepsilon)\, 2\pi a^2} \qquad \sigma_{II}^{free} = \frac{\varepsilon Q}{(\varepsilon_0 + \varepsilon)\, 2\pi a^2}}$$

c)

$$\sigma_{total} = \varepsilon_0 \bar{E} \cdot \hat{n} \Big|_{r=a}$$

$$\sigma_{TOTAL} = \frac{\varepsilon_0 Q}{(\varepsilon_0 + \varepsilon)\, 2\pi a^2}$$ in region I and II

$$\sigma_{polarization} + \sigma_{free} = \sigma_{total}$$

so

vacuum doesn't polarize

$$\boxed{\begin{aligned} \sigma_{pol}^{I} &= \emptyset \\ \sigma_{pol}^{II} &= \frac{(\varepsilon_0 - \varepsilon)\, Q}{(\varepsilon_0 + \varepsilon)\, 2\pi a^2} \end{aligned}}$$

2) (J. 4.12.)

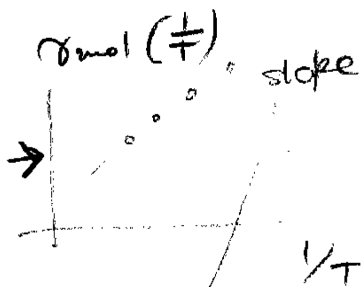
$$\gamma_{mol} = \frac{3}{N} \frac{\left(\frac{\varepsilon}{\varepsilon_0} - 1\right)}{\underbrace{\left(\frac{\varepsilon}{\varepsilon_0} + 2\right)}_{\sim 3 \text{ (from data)}}} \simeq \frac{1}{N}\left[\frac{\varepsilon - \varepsilon_0}{\varepsilon}\right]$$

using ideal gas law

$$\gamma_{mol.} = \frac{kT}{p}\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon}\right) \rightarrow$$

known

known



the microscopical model

$$\gamma_{mol} = \gamma_i + \frac{1}{3\varepsilon_0}\frac{p_0^2}{kT} \dashrightarrow$$

is also

$$\frac{p_0^2}{3\varepsilon_0 k}$$

$$\boxed{p_0 = 6 \times 10^{-30} \text{ C.m}}$$

3) Show $\chi_e = \dfrac{(b^2-a^2)\,\rho g h \ln\left(\frac{b}{a}\right)}{\varepsilon_0 V^2}$

energy in capacitor = $\frac{1}{2}(C_v + C_d)V^2 = W_c$

vacuum dielectric

$$C_v = \frac{2\pi\varepsilon_0(\cancel{H}-h)}{\ln\left(\frac{b}{a}\right)} \qquad C_d = \frac{2\pi\varepsilon h}{\ln\left(\frac{b}{a}\right)}$$

potential energy of fluid = $\dfrac{mgh}{2} = \dfrac{gh^2(b^2-a^2)\pi\rho}{2}$

$$W = W_c + W_g$$

equilibrium $\delta W = 0 \rightarrow \delta W_c = -\delta W_g$

$$\delta W_c = \frac{\delta h}{\cancel{2}} \frac{\cancel{2}\pi}{\ln\left(\frac{b}{a}\right)}(\varepsilon - \varepsilon_0)V^2$$

$$\delta W_g = \delta h \frac{\cancel{2}h(b^2-a^2)\pi\rho}{\cancel{2}}$$

$$\boxed{\chi = \frac{\varepsilon - \varepsilon_0}{\varepsilon_0} = -\frac{(b^2-a^2)\cancel{\pi}\rho h g \ln\left(\frac{b}{a}\right)}{\varepsilon_0 \cancel{2\pi} V^2}}$$

Q.E.D.

4)

$$\bar{A} = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(\bar{r}')}{|\bar{r}' - \bar{r}|} d^3r'$$

$$\frac{1}{4\pi|\bar{r}' - \bar{r}|} = \sum_{\ell m} \frac{1}{2\ell+1} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \varphi) \frac{r_<^\ell}{r_>^{\ell+1}}$$

$$\bar{J} = \bar{\omega} \times \bar{r}\, \delta(r-a)\, \sigma$$

$$J_x = -\omega r \sin\theta\, \delta(r-a)\, \sigma \sin\varphi \quad \to \quad \frac{i}{2}\sqrt{\frac{8\pi}{3}}\left(Y_{1-1} + Y_{11}\right)$$

$$J_y = +\omega r \sin\theta\, \delta(r-a)\, \sigma \cos\varphi \quad \to \quad \frac{1}{2}\left(Y_{1-1} - Y_{11}\right)$$

so;

$$A_x = \mu_0 \sigma \omega \frac{a}{3} \frac{1}{2} \sqrt{\frac{8\pi}{3}} \left(Y_{1-1} - Y_{11}\right) \times \begin{cases} r & r < a \\ \frac{a^2}{r^2} & r > a \end{cases}$$

$\Downarrow$

$$\boxed{\bar{A} = \mu_0 \frac{\sigma \omega}{3} a \sin\theta\, \hat{\varphi} \times \begin{cases} r & r < a \\ \frac{a^3}{r^2} & r > a \end{cases}}$$

$$\boxed{\bar{B} = \bar{\nabla} \times \bar{A} = \begin{cases} \mu_0 \omega Q \frac{2}{3} \hat{z} & (\text{constant}) \\ \mu_0 \omega \sigma \frac{a^4}{3} \left(\frac{2\cos\theta}{r^3} \hat{r} + \frac{\sin\theta}{r^3} \hat{\theta} \right) & (\text{dipole}) \end{cases}}$$

5)

a) $\cancel{[\bar{A}(\bar{B}\times\bar{C})]^2 = [\bar{A}(\bar{B}\times\bar{C})]_i[\bar{A}(\bar{B}\times\bar{C})]_i = A_i(\bar{B}\times\bar{C})_i}$

$$[\bar{A}\cdot(\bar{B}\times\bar{C})]^2 = (\varepsilon_{ijk} A_i B_j C_k)^2$$

$$= \varepsilon_{ijk} A_i B_j C_k \, \varepsilon_{lmn} A_l B_m C_n$$

$$= \underbrace{\varepsilon_{ijk}\varepsilon_{lmn}} A_i B_j C_k A_l B_m C_n$$

$\cancel{\delta_{i1}(\delta_{2j}\delta_{3k} - \delta_{k2}\delta_{j3}) + \delta_{j1}(\delta_{i2}\delta_{k3} - \delta_{k2}\delta_{i3}) + \delta_{k1}(\delta_{i2}\delta_{j3} - \delta_{j2}\delta_{i3})}$

$\cancel{\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{vmatrix}}$

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \\ \delta_{in} & \delta_{jn} & \delta_{kn} \end{vmatrix}$$

$\bar{A}\cdot\bar{A}$, $\bar{B}\cdot\bar{B}$, $\bar{C}\cdot\bar{C}$ $\quad \bar{B}\cdot\bar{A}$, $\bar{C}\cdot\bar{B}$, $\bar{A}\cdot\bar{C}$

$$= A_i B_j C_k A_i B_j C_k + A_i B_j C_k A_j B_k C_i +$$

$\bar{A}\cdot\bar{B}$

$$A_i B_j C_k A_k B_i C_j - A_i B_j C_k A_i B_k C_j$$

$$- A_i B_j C_k A_j B_i C_k - A_i B_j C_k A_k B_j C_i$$

$$[\bar{A}(\bar{B}\times\bar{C})]^2 = \bar{A}^2\bar{B}^2\bar{C}^2 + (\bar{A}\cdot\bar{C})(\bar{B}\cdot\bar{A})(\bar{C}\cdot\bar{B}) + (\bar{A}\cdot\bar{B})(\bar{B}\cdot\bar{C})(\bar{C}\cdot\bar{A})$$

$$- \bar{A}^2(\bar{B}\cdot\bar{C})(\bar{C}\cdot\bar{B}) - (\bar{A}\cdot\bar{B})(\bar{B}\cdot\bar{A})\bar{C}^2 - (\bar{A}\cdot\bar{C})\bar{B}^2(\bar{C}\cdot\bar{A}) =$$

$$\boxed{\begin{aligned}[\bar{A}(\bar{B}\times\bar{C})]^2 &= \bar{A}^2\bar{B}^2\bar{C}^2 + (\bar{A}\cdot\bar{C})(\bar{B}\cdot\bar{A})(\bar{C}\cdot\bar{B}) + (\bar{A}\cdot\bar{B})(\bar{B}\cdot\bar{C})(\bar{C}\cdot\bar{A}) - \\ &- \bar{A}(\bar{B}\cdot\bar{C})^2 - (\bar{A}\cdot\bar{B})^2\bar{C}^2 - (\bar{A}\cdot\bar{C})^2\bar{B}^2\end{aligned}}$$

5) c) Sup. M is in its Jordan (or diagonal form)

$$\det(M-\lambda I) = (a-\lambda)(b-\lambda)(c-\lambda)$$

$$P(\lambda) = -\lambda^3 + (a+b+c)\lambda^2 + (-ac-ab-bc)\lambda + abc$$

$$- \boxed{-1 = \frac{Tr(M^0)}{3}}$$

$$- \boxed{Tr\, M}$$

$$(a+b+c)^2 = 2ab+2ac+2bc + a^2+b^2+c^2$$

$$(Tr\, M)^2 \qquad Tr(M^2)$$

$$- \boxed{(-ac-ab-bc) = \frac{Tr(M^2) - (Tr\, M)^2}{2}}$$

The last term is the more complicated one; but I will use a trick: we know that $P(M) = 0$

so

$$-M^3 + (Tr\, M) M^2 + \frac{Tr(M^2) - (Tr\, M)^2}{2} M + abc = 0$$

taking the trace

$$-Tr(M^3) + (Tr\, M)\, Tr(M^2) + \frac{Tr(M^2) - (Tr\, M)^2}{2} Tr\, M + 3abc = 0$$

so

$$\boxed{abc = \det M = \frac{1}{6}\left[(Tr\, M)^3 - 3\, Tr(M)\, Tr(M^2) + 2\, Tr(M^3)\right]}$$