Problem set #2 Solutions P. 209.

1) a)

$$G(\bar{r};\bar{r}') = \frac{1}{\sqrt{\underbrace{(x-x')^2+(y-y')^2}_{D^2}+(z-z')^2}}$$

$$G(x;y;x';y') = \int_{-Z}^{Z} G(\bar{r},\bar{r}')\, d(z-z')$$

$$= \int_{-Z}^{Z} \frac{1}{\sqrt{D^2+\xi^2}}\, d\xi$$

$$= \ln\left(\xi+\sqrt{D^2+\xi^2}\right)\Big|_{-Z}^{Z}$$

$$= \ln\left(\frac{Z+\sqrt{D^2+Z^2}}{-Z+\sqrt{D^2+Z^2}}\right)$$

$$\underset{D \ll Z}{=} \ln\left(\frac{Z+Z+D^2/Z}{\cancel{-Z}+\cancel{Z}+D^2/Z}\right) \sim \ln\left(\frac{2Z^2}{D^2}\right)$$

$$\boxed{\begin{aligned} G^{2D}(\bar{r};\bar{r}') &= -\ln D^2 = -\ln\left[(x-x')^2+(y-y')^2\right] \\ &= -\ln\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right] \end{aligned}}$$

1) (b) $\nabla'^2 G^{2D}(\bar{r},\bar{r}') = -4\pi\, \delta(\bar{r}-\bar{r}')$

$$\left(\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi'^2}\right)\frac{1}{2\pi}\sum_m e^{im(\phi-\phi')} g_m(\rho;\rho') = -4\pi\frac{\delta(\rho-\rho')\delta(\phi-\phi')}{\rho'}$$

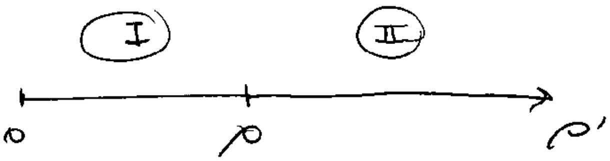
$$\frac{1}{2\pi}\sum_m \left(\frac{1}{\rho'}\frac{1}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} + \frac{1}{\rho^2}m^2\right) g_m(\rho;\rho')e^{im(\phi-\phi')} = \frac{-4\pi\delta(\rho-\rho')}{\rho'}\underbrace{\sum_m \frac{e^{im(\phi-\phi')}}{2\pi}}_{\delta(\phi-\phi')}$$

$$= \sum_m \frac{-4\pi\delta(\rho-\rho')}{\rho'}\frac{e^{im(\phi-\phi')}}{2\pi}$$

$\Downarrow$ (unicity of Fourier series)

$$\boxed{\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} + \frac{1}{\rho'^2}m^2\right] g_m(\rho;\rho') = -4\pi\frac{\delta(\rho-\rho')}{\rho'}}$$

1) c)



for regions I & II

$$\left(\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} - \frac{m^2}{\rho'^2}\right) g_m^{I/II}(\rho;\rho') = 0$$

$$\Downarrow$$

$$\cancel{g_m^{I/II}} \begin{cases} g_m^{I} = A\rho^{|m|} + \cancel{B\rho^{-m}} \\ g_m^{II} = \cancel{C\rho^{m}} + D\rho^{-|m|} \end{cases}$$

well behaved at 0 and ∞

g must be continous at $\rho' = \rho^*$

$$A\rho^{*|m|} = D\rho^{*-|m|}$$

$$\frac{A}{D} = \rho^{*-2|m|}$$

and integrating the original equation

$$\frac{\partial}{\partial\rho'} g \Big|_{\rho-\varepsilon}^{\rho+\varepsilon} = \frac{-4\pi}{\rho}$$

$$D(-|m|)\rho^{-|m|-1} - A|m|\rho^{|m|-1} = -4\pi/\rho$$

$$-\cancel{|m|} D\rho^{-|m|} = \frac{-4\pi}{|m|} + A\cancel{|m|}\rho^{|m|}$$

$$-D = \frac{-4\pi}{|m|}\rho^{|m|} + A\rho^{2m}$$

$$-D = -\frac{4\pi}{|m|}\rho^{|m|} + \cancel{D\rho^{-2|m|}\rho^{2|m|}}$$

$$-2D = -\frac{4\pi}{|m|}\rho^{|m|}$$

$$D = \frac{2\pi\rho^{|m|}}{|m|} \qquad A = \frac{2\pi\rho^{-|m|}}{|m|}$$

final solution

I $\quad g_m(\rho' < \rho) = \frac{2\pi}{|m|}\rho^{-|m|}\rho'^{|m|} = \frac{2\pi}{|m|}\left(\frac{\rho'}{\rho}\right)^{|m|}$

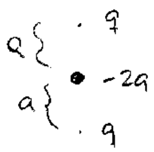
II $\quad g_m(\rho' > \rho) = \frac{2\pi}{|m|}\rho^{|m|}\rho'^{-|m|} = \frac{2\pi}{|m|}\left(\frac{\rho}{\rho'}\right)^{|m|}$

$$\boxed{g_m(\rho,\rho') = \frac{2\pi}{|m|}\left(\frac{\rho_<}{\rho_>}\right)^{|m|}} \xrightarrow[\text{limit } m=0]{\text{in the}} g_0 = 2\pi \ln(\rho_>^2) + \text{inessential constant}$$

$$\downarrow$$

$$\boxed{G(\rho;\rho';\phi;\phi') = -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos[m(\phi-\phi')]}$$

2) a)



$$\Phi(x,y,z) = \frac{q}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{2q}{\sqrt{x^2+y^2+z^2}} + \frac{q}{\sqrt{x^2+y^2+(z+a)^2}}$$

$$= \frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{2q}{r} + \frac{q}{\sqrt{r^2+a^2+2ar\cos\theta}}$$

$$\cancel{1/\sqrt{r^2+a^2-2ar\cos\theta} \sim \frac{1}{\sqrt{r^2-2ar\cos\theta}}}$$

$$\cancel{\left[r^2-2ar\cos\theta\right]^{-1/2}}$$

$$\downarrow$$

$$\cancel{+\tfrac{1}{2}\left[r^2-2ar\cos\theta\right]^{-3/2}(+2r\cos\theta)}$$

Taylor series in a

$$\left[r^2+a^2\mp 2ar\cos\theta\right]^{-1/2}$$

$$\downarrow$$

$$-\frac{1}{2}\left[r^2+a^2\mp 2ar\cos\theta\right]^{-3/2}(2a\mp 2r\cos\theta)$$

$$\downarrow$$

$$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left[r^2+a^2\mp 2ar\cos\theta\right]^{-5/2}(2a\mp 2r\cos\theta)^2 + \left(-\frac{1}{2}\right)\left[r^2+a^2\mp 2ar\cos\theta\right]^{-\frac{3}{2}}(2)$$

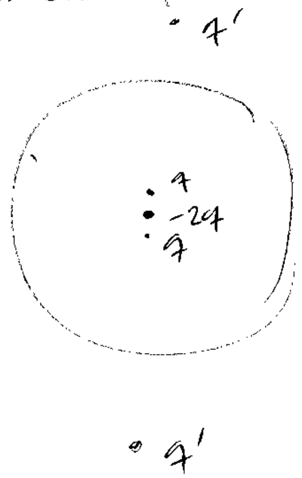
$$\Downarrow$$

$$\left[r^2+a^2\mp 2ar\cos\theta\right]^{-1/2} \sim r^{-1} \pm r^{-3}\,2r\cos\theta + \left[\left(+\frac{3}{4}\right)r^{-5}\,4\,r^2\cos^2\theta - r^{-3}\right]\frac{a^2}{2}$$

$$\phi(x,y,z) = q\left(\cancel{r^{-1}} + r^{-2}\cancel{2a\cos\theta} + r^{-3}(3\cos^2\theta-1)\frac{a^2}{2}\right) - \cancel{2qr^{-1}} + q\left(\cancel{r^{-1}} - r^{-2}\cancel{2a\cos\theta} + r^{-3}(3\cos^2\theta-1)\frac{a^2}{2}\right) =$$

$$\boxed{\Phi = \underset{Q}{a^2q}\, r^{-3}\underset{\propto P_2(\cos\theta)}{\left(3\cos^2\theta-1\right)}} \leftarrow \text{4-pole field.}$$

2) b) Turn-on sphere



2 images + "image" of $-2q$ at infinite = overall increase of potential.

$$\Phi_{images} = -q\left[\frac{b/a}{\left(r^2+\frac{b^2}{a^2}-2\frac{rb^2}{a}\cos\theta\right)^{1/2}} + \frac{\partial/a}{\left(r^2+\frac{b^2}{a^2}-2\frac{rb^2}{a}\cos\theta\right)^{1/2}} - \frac{2}{b}\right]$$

using the same limit as in

$$= +q\,\frac{1}{b}\,\frac{a^2r^2}{b^4}\left(1-3\cos^2\theta\right)$$

this potential must be added to the previous one:

$$\boxed{\Phi_{FINAL} = 2Q\left(1-\frac{r^5}{b^5}\right)P_2(\cos\theta)}$$

gaussian units.

~~4.a)~~ 3 a)

In the convention where all $Q^{(\ell)}$ can be writen as.

$$Q^{(\ell)}_{\underbrace{x_i x_j \cdots}_{\ell \text{ elements}}} = \int \underbrace{x_i x_j \cdots}_{\ell} \rho(\bar{x})\, d^3r \qquad i;j;k \ldots = 1;2 \text{ or } 3$$

translating the origin:

$$\tilde{Q}^{(\ell)}_{x_i x_j \cdots} = \int x_i x_j \; \rho(\bar{x} + \tilde{x})\, d^3r$$

$$= \int (x_i - \tilde{x}_i)(x_j - \tilde{x}_j)(x_k - \tilde{x}_k) \cdots \rho(\bar{x})\, d^3r$$

expanding this product

$$= \int x_i x_j x_k \cdots \rho(\bar{x})\, d^3r \;+ \;\leftarrow Q^{(\ell)}_{x_i x_j \cdots}$$

$$+ \; -\tilde{x}_i \int (x_j x_k \cdots)\, \rho(\bar{x})\, d^3r$$

$$+ \; -\tilde{x}_j \int (x_i x_k \cdots)\, \rho(\bar{x})\, d^3r$$

$$\vdots$$

$$+ \; (\tilde{x}_i \tilde{x}_j \tilde{x}_k \cdots) \int \rho(\bar{x})\, d^3r$$

combinations of $Q^{(\ell')}$ where $\ell' < \ell$ all equal to $\emptyset$

so

$$= Q^{(\ell)}_{x_i x_j \cdots}$$

Q.E.D.

3b)

$P_i = \int x_i \rho(\bar{r})$

$\tilde{P}_i = \int (x_i - \tilde{x}_i)\rho(\bar{r}) = \int x_i \rho(\bar{r}) - \tilde{x}_i \int \rho(r)$

$= p_i - \tilde{x}_i Q$

$$\boxed{\tilde{P} = P - \tilde{R} Q}$$

$r^2 = x_i x_i$

$(x_i - \tilde{x}_i)(x_i - \tilde{x}_i)$

$x_i(x_i - \tilde{x}_i) - \tilde{x}_i(x_i - \tilde{x}_i)$

$x_i x_i - x_i \tilde{x}_i - \tilde{x}_i x_i - \tilde{x}_i \tilde{x}_i$

$x_i x_i - 2 x_i \tilde{x}_i - \tilde{x}_i \tilde{x}_i$

$r^2 - 2\bar{r}\cdot\tilde{R} - \tilde{R}^2$

$Q_{ij} = \int \left(3 x_i x_j - r^2 \delta_{ij}\right)\rho(\bar{r})\, d^3x$

$\tilde{Q}_{ij} = \int \Big[3(x_i - \tilde{x}_i)(x_j - \tilde{x}_j) - (r^2 - 2\bar{r}\cdot\tilde{R} - \tilde{R}^2)\delta_{ij}\Big]\rho(\bar{r})\, d^3r$

$= \int \Big(3 x_i x_j - 3 x_i \tilde{x}_j - 3\tilde{x}_i x_j - 3\tilde{x}_i \tilde{x}_j - r^2\delta_{ij} + 2\bar{r}\cdot\tilde{R}\delta_{ij} + R^2\delta_{ij}\Big)\rho(\bar{r})\, d^3r$

$= \int \left(3 x_i x_j - r^2 \delta_{ij}\right)\rho(\bar{r}) - 3\tilde{x}_j \int x_i \rho(\bar{r}) - 3\tilde{x}_i \int x_j \rho(\bar{r})$

$- 3\tilde{x}_i \tilde{x}_j \int \rho(\bar{r})\, d^3r$

$+ 2\tilde{R}\cdot \int \bar{r}\rho(\bar{r})\delta_{ij} + R^2\delta_{ij}\int \rho(\bar{r})$

$$\boxed{\tilde{Q}_{ij} = Q_{ij} - 3\tilde{x}_j P_i - 3\tilde{x}_i P_j - 3\tilde{x}_i\tilde{x}_j Q + 2\tilde{R}\cdot\bar{P}\delta_{ij} + R^2 Q\delta_{ij}}$$

$\tilde{Q} = Q - 3\bar{P}\otimes\tilde{R} - 3\tilde{R}\otimes\bar{P} - 3R\otimes R Q + 2\tilde{R}\cdot\bar{P}\,\mathbb{1} + R^2 Q\,\mathbb{1}$

3)c) consider $\tilde{R} = \frac{P}{Q}$

for a general Q_{ij} there are 5 parameter to play with an R only gives 3 parameters & in general so the system is over-constrained has no sol.

4) a) $W = \int \rho \Phi \qquad \Phi = \Phi_0 + \partial_i \phi\, x_i + \frac{1}{2} \partial_i \partial_j \phi\, x_i x_j$

$$W = \int \rho \phi_0 + \int \rho\, \partial_i \phi\, x_i + \frac{1}{2} \int \rho\, \partial_i \partial_j \phi\, x_i x_j$$

$$= \phi_0 \int \rho + \partial_i \phi \int \rho x_i + \frac{1}{2} \partial_i \partial_j \phi \int \rho x_i x_j + \cdots$$

$$W(\bar{x}) = \phi(\bar{x})\, q + \partial_i \phi(\bar{x})\, p_i + \frac{1}{6} \partial_i \partial_j \phi \int \rho (3 x_i x_j - \delta_{ij} r^2)$$

$$+ \cancel{\underbrace{\underbrace{\delta_{ij}\, \partial_i \partial_j \phi}_{\partial_i \partial_i \phi}}_{\nabla^2 \phi = 0} \int \rho r^2}$$

(*)

$$\boxed{W(\bar{x}) = \phi(\bar{x})\, q + \partial_i \phi(\bar{x})\, p_i + \frac{1}{6} \partial_i \partial_j \phi(\bar{x})\, Q_{ij} + \cdots}$$

$$W(\bar{x}) = q \phi(\bar{x}) - \bar{E} \cdot \bar{p} - \frac{1}{6} \partial_i E_j Q_{ij} + \cdots$$

$$\bar{F} = -\bar{\nabla} W$$

$$\boxed{\bar{F} = \underbrace{-\bar{\nabla} \phi(\bar{x})}_{\bar{E}} q + \bar{\nabla}(\bar{E} \cdot \bar{p}) + \frac{1}{6} \bar{\nabla} \left[\partial_i E_j Q_{ij} \right]}$$

(*) is still valid but now rotations can change the momenta (for example rotating the dipole moment)

$$q \longrightarrow q \quad \text{(scalar)}$$

$$\bar{p} \longrightarrow \bar{p} + \hat{x}_1 \times \bar{p}\, \delta\theta$$

$$p_i \longrightarrow p_i + \underbrace{\varepsilon_{i1k}\, p_k\, \delta\theta}_{\delta p_i}$$

$$Q_{ij} \longrightarrow Q_{ij} + (\varepsilon_{i1k} Q_{kj} + \varepsilon_{j1k} Q_{ki})\,\delta\theta$$

so:

$$\underbrace{\delta W_1 = \Big\{ \partial_i\phi\, \varepsilon_{i1k} p_k + \frac{1}{6}\, \partial_i\partial_j\phi \Big(\underbrace{\varepsilon_{i1k} Q_{kj}}_{Q_{2j}\delta_{i3} - Q_{3j}\delta_{i2}} + \underbrace{\varepsilon_{j1k} Q_{ki}}_{Q_{2i}\delta_{j3} - Q_{3i}\delta_{j2}} \Big) \Big\}\, \delta\theta}_{-N_1}$$

$$N_1 = -[\bar{p} \times \bar{\nabla}\phi]_1 - \frac{1}{6}\Big\{ \partial_i\partial_j\phi\, Q_{2j}\delta_{i3} - \partial_i\partial_j\phi\, Q_{3j}\delta_{i2} + \partial_i\partial_j\phi\, Q_{2i}\delta_{j3} - \partial_i\partial_j\phi\, Q_{3i}\delta_{j2} \Big\}$$

same

$$= -[\bar{p} \times \bar{\nabla}\phi]_1 - \frac{1}{6}\Big\{ \partial_3\partial_j\phi\, Q_{2j} - \partial_2\partial_j\phi\, Q_{3j} + \partial_i\partial_3\phi\, Q_{2i} - \partial_i\partial_2\phi\, Q_{3i} \Big\}$$

same.

$$\boxed{N_1 = [\bar{p} \times \bar{E}] + \frac{1}{3}\Big\{ \partial_3 E_j Q_{2j} - \partial_2 E_j Q_{3j} \Big\}}$$

This is an elegant way, starting with $\bar{N} = \int \rho\, r \times E(r)$ is also correct.

5) From 4)

5) $W = -\frac{1}{6}\left(\partial_i E_j Q_{ij}\right) \longrightarrow$ by symmetry $\partial_x E_y = -\partial_y E_x$

and $\partial_x E_z = 0$, $\partial_y E_z = 0$

and $\partial_z E_x = 0$, $\partial_z E_y = 0$

so off-diagonal elements of Q cancel each other.

$\bar{\bar{Q}} = \begin{pmatrix} -eQ/2 & 0 & 0 \\ 0 & -eQ/2 & 0 \\ 0 & 0 & eQ \end{pmatrix} \longleftarrow$ "is like"

$$W = -\frac{1}{6}\left[-\partial_x E_x \frac{eQ}{2} - \partial_y E_y \frac{eQ}{2} + \partial_z E_z eQ\right]$$

(Jackson is not very clear)

$$= -\frac{e}{6}\left[-\left(\partial_x E_x + \partial_y E_y\right)\frac{Q}{2} + \partial_z E_z Q\right]$$

$\bar{E}$ is axially symmetrical $\Rightarrow \partial_x E_x = \partial_y E_y$

$\bar{\nabla}\cdot\bar{E} = 0 \Rightarrow \partial_x E_x + \partial_y E_y + \partial_z E_z = 0$

$\Downarrow$

$$\partial_x E_x = \partial_y E_y = -\frac{\partial_z E_z}{2}$$

$$= -\frac{e}{6}\left[-\left(-\partial_z E_z\right)\frac{Q}{2} + \left(\partial_z E_z\right)Q\right]$$

$$= -\frac{e}{6}\,\frac{3}{2}\left(\partial_z E_z\right)Q$$

$$\boxed{W = -\frac{e}{4}\,Q\left(\partial_z E_z\right)}$$

5) b)

$$Q = 2\times10^{-28}\ m^2$$

$$W/h = 10\ \mathrm{MHz}$$

$$W = -\frac{e}{4} Q \left(\frac{\partial E}{\partial z}\right)_0$$

$$\left(\frac{\partial E}{\partial z}\right)_0 = -\frac{4W}{eQ} = \boxed{-0.08\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)}$$

5) c)

$$eQ = Q_{33} = \int (3z^2 - r^2)\rho(r)\, d^3x$$

$$= \rho \int_0^{2\pi} d\varphi \int_{z=-b}^{b} dz \int_0^{\frac{a}{b}\sqrt{b^2-z^2}} (2z^2 - r^2)\, r\, dr$$

$$= \rho\, 2\pi \int_{-b}^{b} dz \; \frac{\cancel{2}z^2 r^2}{\cancel{2}} - \frac{1}{4} r^4 \Big|_0^{\frac{a}{b}\sqrt{b^2-z^2}}$$

$$= \rho\, 2\pi \int_{-b}^{b} dz \left\{ \cancel{2} z^2 \frac{a^2}{b^2}(b^2 - z^2) - \frac{1}{4}\frac{a^4}{b^4}(b^2 - z^2)^2 \right\}$$

$$= \rho\, 2\pi \int_{-b}^{b} dz \; \frac{z^2 a^2 \cancel{b^2}}{\cancel{b^2}} - z^4 \frac{a^2}{b^2} - \frac{1}{4}\frac{a^4}{b^4} b^4 + \frac{a^4}{4b^4} 2b^2 z^2 - \frac{1}{4}\frac{a^4}{b^4} z^4$$

$$eQ = \rho\, 2\pi \frac{4}{15} a^2 b (b^2 - a^2) \leftarrow \left(\rho = Ze \Big/ \tfrac{4}{3}\pi a^2 b\right)$$

$$Q = \frac{2}{5} Z (a+b)(a-b) \Rightarrow \boxed{\frac{a-b}{b} \simeq 0.1}$$