HW12

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J# 14.26

1)

$E = 10^{13}\,eV$

$B \sim 10^{-4}$ Gauss

a) ~~$E = 10^{13}\,e$~~

$c\,P_{(\perp)} = eB\rho$

$E \gg m_e$ we can take $\rho c = E \rightsquigarrow \rho = \frac{E}{eB}$

$$\boxed{\rho = 10^{12}\,m}$$

$$\boxed{\omega_0 = \frac{c}{\rho} = 3\times10^{-4}\,s^{-1}}$$

$$\boxed{\omega_c = \frac{3}{2}\omega_0\gamma^3 = 4\times10^{18}\,s^{-1}}$$

$$\boxed{\hbar\omega_c = 0.002\ MeV}$$

b) Power spectrum

$$\frac{dI}{d\omega} \sim \frac{e^2}{c}\left(\frac{\omega \rho}{c}\right)^{1/3}$$

for $\omega \ll \omega_c$ $\quad \frac{dI}{d\omega} \simeq \sqrt{\frac{3\pi}{2}}\,\frac{e^2}{c}\,\gamma\left(\frac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c}$

for $\omega \gg \omega_c$ and $\quad P \propto \omega_0 \frac{dI}{d\omega}$

$$P \propto \begin{cases} \omega_0 (\omega \rho)^{1/3} & ; \quad \omega \ll \omega_c \\ \omega_0 \gamma \left(\frac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c} & ; \quad \omega \gg \omega_c \end{cases}$$

$$\rho = \frac{E}{eB} \propto E \quad ; \quad \omega_0 = \frac{c}{\rho} \propto \frac{1}{E}$$

$$\Rightarrow \omega_0 (\omega \rho)^{1/3} \propto \frac{1}{E}(\omega E)^{1/3} \propto \left(\frac{\omega}{E^2}\right)^{1/3} \Rightarrow$$

$$\Rightarrow P \propto \left(\frac{\omega}{E^2}\right)^{1/3} \quad (\omega \ll \omega_c)$$

$$\omega_0 \propto \frac{1}{E} \; ; \; \gamma \propto E$$

$\omega\gamma \propto 1$ (no dependence)

$$\Rightarrow \; P \propto \left(\frac{\omega}{\omega_0}\right)^{1/2} e^{-\omega/\omega_c} \qquad (\omega \gg \omega_c)$$

But $\omega_c = \frac{3}{2}\left(\frac{eB}{mc}\right)\left(\frac{E}{mc^2}\right)^2 \cos\theta \;\; \propto E^2$

$$\Rightarrow \left(\frac{\omega}{\omega_c}\right) \propto \left(\frac{\omega}{E^2}\right) \;\Rightarrow\; P \propto \left(\frac{\omega}{E^2}\right)^{1/2} e^{-\omega/\omega_c} \qquad (\omega \gg \omega_c)$$

$$\omega_c = \frac{3}{2}\left(\frac{eB}{mc}\right)\left(\frac{E}{mc^2}\right)^2 \cos\theta \propto E^2 \;\Rightarrow\; \frac{\omega}{\omega_c} \propto \frac{\omega}{E^2}$$

$$\Rightarrow \; P \propto \left(\frac{\omega}{E^2}\right)^{1/2} e^{-\omega/\omega_c} \qquad \omega \gg \omega_c$$

$$P \propto \left(\frac{\omega}{E^2}\right)^{1/3} \begin{cases} 1 & (\omega \ll \omega_c \\ \left(\frac{\omega}{E^2}\right)^{1/6} e^{-\omega/\omega_c} & (\omega \gg \omega_c) \end{cases}$$

But $\frac{\omega}{E^2} \propto \frac{\omega}{\omega_c}$ $\quad P \propto \left(\frac{\omega}{E^2}\right)^{1/3} f\left(\frac{\omega}{\omega_c}\right)$ $\quad f(0) \sim 1$, $f(x) \sim \exp(x)$

c) Electron energy spectrum $N(E)dE \propto E^{-n}dE$

$$\langle P(\omega)\rangle = \int_0^\infty dE\, N(E) P(E;\omega) \propto \int_0^\infty E^{-n} dE\, P(E;\omega) \propto$$

$$\propto \int_0^\infty dE\, E^{-n} \left(\frac{\omega}{E^2}\right)^{1/3} f(\omega/\omega_c) \propto$$

$$\propto \omega^{1/3} \int_0^\infty dE\, E^{-(n+2/3)} f(\omega/\omega_c)$$

$f\left(\frac{\omega}{\omega_c}\right)$ dies exponentially after ω_c

$$\omega_c = \frac{3}{2}\frac{eB}{mc}\left(\frac{E}{mc^2}\right)^2 \cos\theta = kE^2$$

so $\omega < \omega_c \Rightarrow \frac{\omega}{\omega_c} < 1 \mapsto \frac{\omega}{kE^2} < 1 \rightarrow E > \sqrt{\frac{\omega}{k}}$

$$\langle P(\omega)\rangle \propto \omega^{1/3} \int_{\sqrt{\frac{\omega}{k}}}^\infty dE\, E^{-(n+2/3)}$$

$$\propto \omega^{1/3} \left[E^{-(n-\frac{1}{3})}\right]_{\sqrt{\frac{\omega}{k}}}^\infty \propto$$

$$\propto \omega^{1/3} \left(\sqrt{\frac{\omega}{k}}\right)^{-(n-1/3)} \propto \omega^{1/3} \omega^{-n/2} \omega^{1/6}$$

$$\Rightarrow \langle P(\omega)\rangle \propto \omega^{-(n-1)/2} \Rightarrow \boxed{\langle P(\omega)\rangle d\omega \propto \omega^{-\alpha} d\omega \quad \alpha \equiv \frac{n-1}{2}}$$

d) $\omega \sim 10^{8}\,s^{-1}$ to $\omega \sim 10^{15}\,s^{-1}$

$\alpha \cong 0.35$ for $\omega > 10^{18}\,s^{-1}$; $\alpha \gtrsim 1.5$

$$w_c = \frac{3}{2}\left(\frac{eB}{mc}\right)\left(\frac{E}{mc^2}\right)^2 \underbrace{\cancel{\cos\theta}}_{1} = \frac{3}{2}\left(\frac{eB}{mc}\right)\left(\frac{E}{mc^2}\right)^2 = kE^2$$

$$k = \frac{3eB}{2m^3c^5}$$

$$\Rightarrow E_c = \sqrt{\frac{w_c}{k}} = \sqrt{\frac{2m^3c^5 w_c}{3eB}} = \sqrt{\frac{2\,(mc^2)^2 w_0}{3\left(\frac{e}{mc}\right)B}} =$$

$$= 6 \times 10^{6}\ \text{MeV}$$

$$n = 2\alpha + 1 \Rightarrow 2(0.35) + 1 = 1.7$$

$$\boxed{n = 1.7\ ;\ E_{max} = 6 \times 10^{6}\ \text{MeV}}$$

e) from b)

$$t \simeq \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right)$$

$$t_{1/2} \simeq \frac{3m^3c^5}{2q^4B^2\gamma_0}$$

if $q = e$; $\gamma_0 = \frac{E}{mc^2}$ $\Rightarrow$ $t_{1/2} \simeq \frac{3c^3}{2B^2E\left(\frac{e}{mc}\right)^4}$

if $B = 10^{-3}$ gauss ; $E = 16$ eV

$\cancel{t_{1/2} \simeq 8300 \text{ years } \frac{E}{16e}}$

from part a) $E = 10^{13}$ eV ; $B = 3\times10^{-4}$ Gauss

$\boxed{t_{1/2} \simeq 9.2 \text{ years}}$ $\ll$ 1000 year (of crab nebula)