HW # 11

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1) a) Show that

$$+e\left\{ \cancel{\frac{\Delta x^\mu \Delta x^\nu}{(\Delta x\cdot u)^2}} + \frac{\Delta x^\mu u^\nu - u^\mu \Delta x^\nu}{(\Delta x\cdot u)^3}\right\} \overset{?}{=} F^{\mu\nu}_{velocity}$$

is the velocity field.

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \Rightarrow F^{i0} = E_i$$

$$F^{i0}_{velocity} = e\frac{\Delta x^i u^0 - u^i \Delta x^0}{(\Delta x^0 u^0 - \Delta x^j u^j)^3} = E_i$$

because $(\Delta x\cdot\Delta x) = 0$

$$u^0 = \gamma \qquad \Delta x^0 = |\bar{\Delta x}| =$$

$$= e\frac{\bar{\Delta x}\,\gamma - \bar{u}\,|\bar{\Delta x}|}{(|\Delta x|\gamma - \bar{\Delta x}\cdot\bar{u})^3} = \frac{e\gamma(\bar{R} - R\bar{\beta})}{[(R - \bar{R}\cdot\bar{\beta})\gamma]^3}$$

$$\left(\begin{matrix}\bar{u} = \gamma\bar{\beta} \\ \bar{\Delta x} = \bar{R}\end{matrix}\right)$$

so

$$F^{i0}_{velocity} = \frac{e \gamma R (\hat{n} - \bar{\beta})}{R^3 \gamma^3 \cancel{\ } (1 - \hat{n} \cdot \bar{\beta})^3} = \underbrace{\frac{e (\hat{n} - \bar{\beta})}{R^2 \gamma^2 (1 - \hat{n} \cdot \bar{\beta})}}$$

but thisis $\bar{E}_{velocity}$! Q.E.D.

The do the same with $(F^{32}; F^{13}; F^{21}) = \bar{B}_{velocity}$

The physical argument is that the rest MUST BE due to acceleration.

1)

b)

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\mu\alpha} F_{\alpha}{}^{\nu} - g^{\mu\nu} \underbrace{F^{\alpha\beta} F_{\alpha\beta}}$$

$\propto E^2 - B^2 = 0$ for radiation.

$$F^{\mu\alpha} F_{\alpha}{}^{\nu} =$$

$$= e\left\{ \frac{\Delta x^{\mu} b^{\alpha} - b^{\mu} \Delta x^{\alpha}}{(\Delta x \cdot u)^2} - \frac{\Delta x^{\mu} u^{\alpha} - u^{\mu} \Delta x^{\alpha}}{(\Delta x \cdot u)^3} (\Delta x \cdot b) \right\} \cdot F_{\alpha}{}^{\nu}$$

$$= e^2 \left\{ \frac{(\Delta x^{\mu} b^{\alpha} - b^{\mu} \Delta x^{\alpha})(\Delta x_{\alpha} b^{\nu} - b_{\alpha} \Delta x^{\nu})}{(\Delta x \cdot u)^4} + \right. \quad ①$$

$$- \frac{(\Delta x^{\mu} b^{\alpha} - b^{\mu} \Delta x^{\alpha})(\Delta x_{\alpha} u^{\nu} - u_{\alpha} \Delta x^{\nu})}{(\Delta x \cdot u)^5} (\Delta x \cdot b) + \quad ②$$

$$- \frac{(\Delta x^{\mu} u^{\alpha} - u^{\mu} \Delta x^{\alpha})(\Delta x_{\alpha} b^{\nu} - b_{\alpha} \Delta x^{\nu})}{(\Delta x \cdot \mu)^5} (\Delta x \cdot b) + \quad ③$$

$$\left. + \frac{(\Delta x^{\mu} u^{\alpha} - u^{\mu} \Delta x^{\alpha})(\Delta x_{\alpha} u^{\nu} - u_{\alpha} \Delta x^{\nu})}{(\Delta x \cdot u)^6} \right\} \quad ④$$

$$① = \frac{\Delta x^\mu b^\alpha \Delta x_\alpha b^\nu - \Delta x^\mu b^\alpha b_\alpha \Delta x^\nu - b^\mu \underbrace{\Delta x^\alpha \Delta x_\alpha}_{0} b^\nu + b^\mu \Delta x^\alpha b_\alpha \Delta x^\nu}{(\Delta x \cdot u)^4}$$

$$② \frac{\left(- \Delta x^\mu b^\alpha \Delta x_\alpha u^\nu + \Delta x^\mu b^\alpha u_\alpha \Delta x^\nu + b^\mu \Delta x^\alpha \Delta x_\alpha u^\nu - b^\mu \Delta x^\alpha u_\alpha \Delta x^\nu\right)(\Delta x \cdot b)}{(\Delta x \cdot u)^5}$$

$\vdots$

use $(b \cdot u) = 0$; $(u \cdot u) = 0$; $(\Delta x \cdot \Delta x) = 0$

$\vdots$

$$\boxed{T^{\mu\nu} = \frac{-e^2}{4\pi (\Delta x \cdot u)^6} \left[(\Delta x \cdot b)^2 c^2 + (b \cdot b)(\Delta x \cdot u)^2 \right] \Delta x^\mu \Delta x^\nu}$$

1) c)

$$\int du\, dv\, dw\, \varepsilon_{\mu\nu\alpha\beta} \frac{\partial x^\mu}{\partial u} \frac{\partial x^\nu}{\partial v} \frac{\partial x^\alpha}{\partial w} K^\beta = S$$

the parametrization is

$$x^\mu = (0; u; v; w)$$

so

$$\frac{\partial x^\mu}{\partial u} = (0; 1; 0; 0) = \delta^\mu{}_1$$

$$\frac{\partial x^\nu}{\partial v} = (0; 0; 1; 0) = \delta^\nu{}_2$$

$$\frac{\partial x^\alpha}{\partial w} = (0; 0; 0; 1) = \delta^\alpha{}_3$$

so

$$S = \int du\, dv\, dw\, \varepsilon_{\mu\nu\alpha\beta}\, \delta^\mu{}_1 \delta^\nu{}_2 \delta^\alpha{}_3 K^\beta =$$

$$= \int du\, dv\, dw\, \underbrace{\varepsilon_{123\beta} K^\beta}_{\varepsilon_{1230} K^0} = \int du\, dv\, dw\, K^0(u; v; w) = \boxed{\int d^3\bar{x}\, K^0(\bar{x})}$$

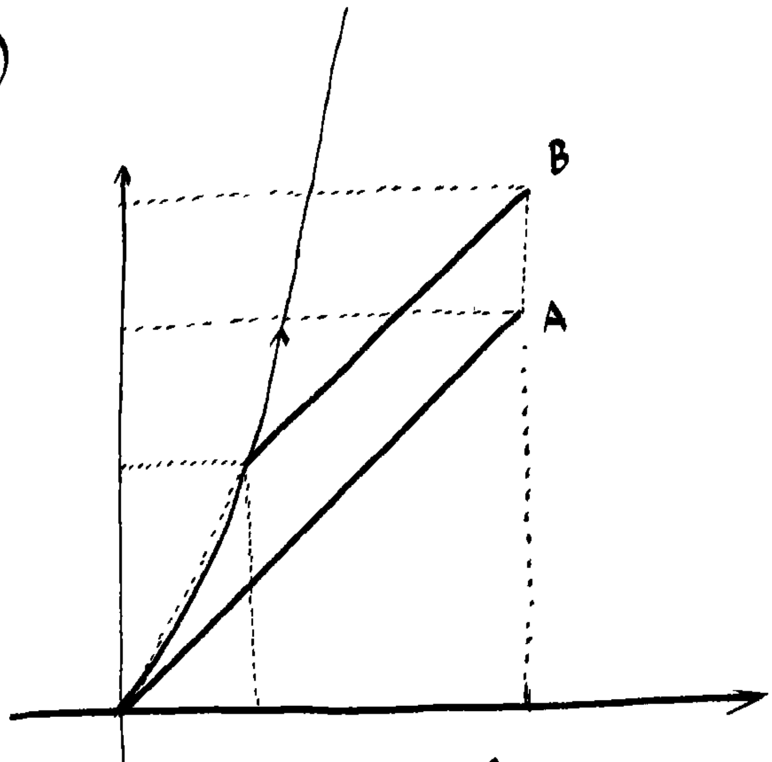
because $\varepsilon_{1231} = \varepsilon_{1232} = \varepsilon_{1233} = 0$

d)

$$K^{\mu} = \frac{-e^2}{4\pi}\left[(b\cdot b)(\Delta x\cdot u)^2 + (\Delta x\cdot b)^2\right]\frac{(\Delta x\cdot k)}{(\Delta x\cdot u)^4}\,\Delta x^{\mu}$$

any 4vector normal to the surface is
normal to Δx and then it is normal to K^{μ}.

e)

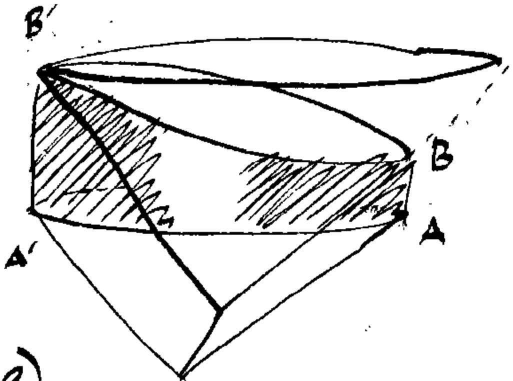


if $\bar{\beta} \perp \hat{n} \Rightarrow \Delta t = \Delta t'$ + second order corrections

if $\bar{\beta} \parallel \hat{n} \rightarrow \Delta t = \Delta t' - \underbrace{\Delta t' \beta}_{\text{distance traveled by source}} = \Delta t'(1 - \beta)$

so in the general case it is likely that

$$\boxed{\Delta t = \Delta t'(1 - \hat{n} \cdot \bar{\beta})} \quad \frac{dt}{dt'} = 1 - \hat{n} \cdot \bar{\beta}$$



e)

Intergral over sphere $(u; v; w) = (\theta; \varphi; t)$

$$\frac{dx^\mu}{d\theta} = \begin{pmatrix} 0 \\ r\hat{\theta} \end{pmatrix} ; \quad \frac{dx^\mu}{d\varphi} = \begin{pmatrix} 0 \\ r\sin\theta\,\hat{\varphi} \end{pmatrix} ; \quad \frac{dx^\mu}{dt} = \begin{pmatrix} 1 \\ \bar{0} \end{pmatrix}$$

$$\Rightarrow K\varepsilon_{\mu\nu\alpha\beta}\frac{dx^\mu}{du}\frac{dx^\nu}{dv}\frac{dx^\alpha}{dw} = \begin{vmatrix} 0 & 0 & 1 & K^0 \\ r\hat{\theta} & r\sin\theta\,\hat{\varphi} & \bar{0} & \bar{K} \end{vmatrix} = (\bar{K}\cdot\hat{r})\, r^2\sin\theta$$

$$\int_S K^\mu n_\mu = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \int_t^{t+\Delta t(\theta;\varphi)} dt \; r^2 \sin\theta\,(\bar{K}\cdot\hat{r})$$

$$= \int r^2 d\Omega\,(\bar{K}\cdot\hat{r})\underset{\Delta t'(1-\hat{n}\cdot\bar{\beta})}{\underset{\parallel}{\Delta t}} = \Delta t' \int r^2 d\Omega\,(1-\hat{n}\cdot\bar{\beta})(\bar{K}\cdot\hat{r})$$

$$K^\mu = T^{\mu\nu}k_\nu \; ; \quad k_\nu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \bar{K} = \begin{pmatrix} T^{10} \\ T^{20} \\ T^{30} \end{pmatrix} = \text{energy flux.}$$

$\Downarrow$

$\bar{K}\cdot\hat{r}$ energy flux crossing sphere so

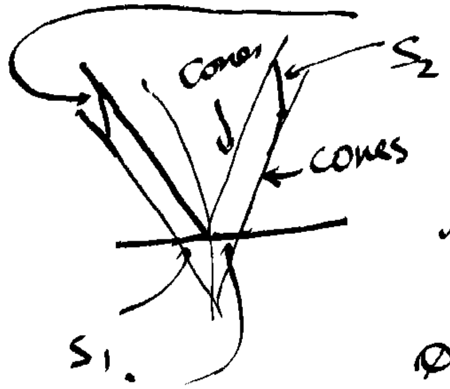
$$\left(1-\hat{n}\cdot\bar{\beta}\right)\left(\bar{k}\cdot\hat{r}\right)$$

$\Rightarrow$ if $k_\nu = \begin{pmatrix} 1 \\ \bar{0} \end{pmatrix}$ this integral represents the total energy ~~flux~~ radiated by particle in unit retarded time.

for $k_\nu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ we get

total momentum component radiated ~~in~~

f) Consider a 3D 4-surface



orientation

$$\int_F K^\mu = \int_{S_1} - \int_{S_2} + \underbrace{\int_{\text{cones}} K^\mu}_{\text{shown to be zero}}$$

$\overset{\text{"}}{\emptyset}$ because conservation

$\Downarrow$

$$\boxed{\int_{S_1} K^\mu = \int_{S_2} K^\mu}$$

same flux of light on both surfaces.

g) In a frame where the particle is at rest

$$T^{\mu\nu} = \frac{-e^2}{4\pi(\Delta x\cdot u)^6}\left[(\Delta x\cdot b)^2c^2 + (b\cdot b)(\Delta x\cdot u)^2\right]\Delta x^\mu \Delta x^\nu$$

$$k^\mu = T^{\mu\nu}k_\nu$$

$$K^\mu = \frac{-e^2}{4\pi(\Delta x\cdot u)^6}\left[(\Delta x\cdot b)^2c^2 + (b\cdot b)(\Delta x\cdot u)^2\right](\Delta x\cdot k)\,\Delta x^\mu$$

$$k^\mu = \begin{pmatrix} k^0 \\ \bar{k} \end{pmatrix};\ \Delta x^\mu = \begin{pmatrix} r \\ r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix};\ u^\mu = \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix};\ b^\mu = \begin{pmatrix} \dot{\gamma}c \\ \bar{a} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{a} \end{pmatrix}$$

(rest frame)

must be perpendicular to u^μ

$$(\Delta x\cdot u) = rc$$

$$(b\cdot b) = -a^2$$

$$(\Delta x\cdot b) = -\vec{r}\cdot\vec{a}$$

$$(\Delta x\cdot k) = rk^0 - \bar{r}\cdot\bar{k}$$

$$K^\mu = \frac{-e^2}{4\pi r^6c^6}\left[(\bar{r}\cdot\bar{a})^2c^2 - a^2r^2c^2\right](rk^0 - \bar{r}\cdot\bar{k})\,\Delta x^\mu$$

We want the integral at fixed time so

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_{r_1}^{r_2} dr\,(r^2 \sin\theta)\, K^0(r, \theta, \varphi) =$$

radius of the two surfaces

$$= \frac{-e^2}{4\pi c^2} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_{r_1}^{r_2} dr \; r^2 \sin\theta \, \frac{1}{r^6} \left[(\bar{r}\cdot\bar{a})^2 - a^2 r^2 \right] (r k^0 - \hat{r}\cdot\bar{k})\, r$$

$$= \frac{-e^2}{4\pi c^4} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_{r_1}^{r_2} dr \left[(\hat{n}\cdot\bar{a})^2 - a^2 \right] (k^0 - \hat{n}\cdot\bar{k}) \sin\theta$$

let choose a direction for $\bar{a} \parallel \hat{z}$ $\quad \bar{a} = \hat{z} a$

$$= \frac{-e^2}{4\pi c^4} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_{r_1}^{r_2} dr \left[a^2 \cos^2\theta - a^2 \right] (k^0 - \hat{n}\cdot\bar{k}) =$$

$$= \frac{a^2 e^2}{4\pi c^4} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta \, d\theta \int_{r_1}^{r_2} dr \, (k^0 - \hat{n}\cdot\bar{k}) =$$

$$= \frac{a^2 e^2}{4\pi c^4} \left\{ \left(\int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta \, d\theta \int_{r_1}^{r_2} dr \right) k^0 - \left(\int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta \, d\theta \int_{r_1}^{r_2} dr \, \hat{n} \right) \cdot \vec{k} \right\}$$

$$\int_0^{2\pi} d\varphi\, \hat{n} = \begin{pmatrix} \sin\theta \int_0^{2\pi} d\varphi \cos\varphi \\ \sin\theta \int_0^{2\pi} d\varphi \sin\varphi \\ \cos\theta \int_0^{2\pi} d\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2\pi\cos\theta \end{pmatrix}$$

and $\cancel{\int_0^{\pi} \sin^3\theta \, d\theta \, d\theta}$

$$\int_0^{\pi} \sin^2\theta \underbrace{\cos\theta\, d\theta}_{d(\sin\theta)} = 0$$

$$\int_0^{2\pi} d\varphi = 2\pi \; ; \; \int_{r_1}^{r_2} dr = (r_2 - r_1)$$

$$\int_0^{\pi} \sin^3\theta\, d\theta = \left[\frac{-3\cos\theta}{4} + \frac{\cos 3\theta}{12} \right]_0^{\pi} = \frac{4}{3}$$

$$\int_{CDC'D'} K^{\mu} n_{\mu} = \frac{2e^2}{3c^4} a^2 (r_2 - r_1) k^0$$

and $(r_2 - r_1) = c\Delta\tau$

$$\int K^{\mu} n_{\mu} = \frac{2e^2}{3c^3} a^2 \Delta\tau\, k^0$$

$$\boxed{\int_{CDC'D'} K^{\mu} n_{\mu} = -\frac{2e^2}{3c^4} (b\cdot b)\, \Delta\tau\, (k\cdot u)}$$

h) $dE = \int_{ABA'B'} K^\mu n_\mu = \int_{CDC'D} K^\mu n_\mu = \frac{-2e^2}{3c^4}(b\cdot b)\underbrace{\Delta t}_{\frac{\Delta t'}{\cancel{\gamma}}}\underbrace{(k\cdot u}_{\omega^0 = \gamma c}$

since $k^\mu = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$$\cancel{dt'}\frac{dE}{dt'} = \frac{-2e^2}{3c^4}(b\cdot b)\cancel{\Delta}\frac{\cancel{\Delta t'}}{\cancel{\gamma}}\cancel{\gamma}c$$

$$\frac{dE}{dt'} = \frac{-2e^2}{3c^3}(b\cdot b)$$

$$(b\cdot b) = \frac{1}{m}\frac{dp}{d\tau}\cdot\frac{1}{m}\frac{dp}{d\tau} = \frac{1}{m^2}\left(\frac{dp}{d\tau}\cdot\frac{dp}{d\tau}\right)$$

for $\gamma \sim 1$ $\quad b^\mu \simeq \begin{pmatrix}0\\ a\end{pmatrix}$ $\cdot$ $\cancel{(b\cdot b)}$

$$(b\cdot b) \sim -a^2$$

$$\boxed{\frac{dE}{dt'} = \frac{2}{3}\frac{e^2a^2}{c^3}}$$

Larmor formula

2) J#14.8

Lorentz force

$$m b^\mu = Ze F^\mu{}_\nu u^\nu \qquad b^\mu = \frac{du^\mu}{d\tau}$$

$$b^\mu = \frac{Ze}{m} F^\mu{}_\nu u^\nu$$

$$u^\nu = \gamma \begin{pmatrix} 1 \\ v \\ 0 \\ 0 \end{pmatrix} \rightarrow b^\mu = \frac{Ze\gamma}{m}\left(F^\mu{}_0 + vF^\mu{}_1\right)$$

$$b^\mu = \frac{e\gamma}{m}\begin{pmatrix} vE_x \\ E_x \\ E_y - vB_z \\ E_z + vB_y \end{pmatrix}$$

$$\bar{B} = 0 \qquad \bar{E} = \frac{Ze}{(x^2+b^2)^{3/2}}\left(x\hat{x} - b\hat{y}\right)$$

$$\begin{cases} E_x = Zex/(x^2+b^2)^{3/2} \\ E_y = -Zeb/(x^2+b^2)^{3/2} \\ E_z = 0 \end{cases}$$

$$\text{so} \quad b^\mu = \frac{Ze^2\gamma}{m(x^2+b^2)^{3/2}}\begin{pmatrix} vx \\ x \\ -b \\ 0 \end{pmatrix}$$

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$$\frac{dp^\mu}{d\tau} = \frac{z Z e^2 \gamma}{(x^2+b^2)^{3/2}} \begin{pmatrix} vx \\ x \\ -b \\ 0 \end{pmatrix} = m b^\mu$$

$$P = -\frac{2}{3}\frac{z^2 e^2}{m^2}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) = \cancel{-\frac{2}{3}\frac{z^2e^2}{m^2}(\ldots)}$$

$$= \frac{-2}{3}\frac{z^2 e^2}{m^2}\left((vx)^2 - x^2 - b^2\right)\frac{z^2 Z^2 e^4 \gamma^2}{(x^2+b^2)^3}$$

$$= \frac{2 z^4 Z^2 e^6 \gamma^2}{3m^2(x^2+b^2)^3}\left[b^2 + x^2 \underbrace{(1-v^2)}_{\frac{1}{\gamma^2}}\right] =$$

$$P = \frac{2 z^4 Z^2 e^6}{3m^2(x^2+b^2)^3}\left[x^2 + \gamma^2 b^2\right] \Big/ c^3$$

c velocity of light.

$$\boxed{\Delta W = \int_{-\infty}^{\infty} dt\, P = \frac{\pi z^4 Z^2 e^6}{4m^2 c^4 \beta}\left(\gamma^2 + \frac{1}{3}\right)\frac{1}{b^3}}$$

J# 14.9 (w/o d)

3) a)

Total energy radiated per unit time

from eq. of motion

$$\frac{dp^\mu}{d\tau} = \frac{q}{c} F^\mu{}_\nu u^\nu$$

$B_z \neq 0$ $F^1_2 = -F^2_1 = B$ other component $= 0$.

$$\frac{dp^\mu}{d\tau} = \frac{q}{c}\begin{pmatrix} 0 \\ \beta u^2 \\ -\beta u^1 \\ 0 \end{pmatrix} ; \quad \begin{matrix} u_1 = \gamma v_x \\ u_2 = \gamma v_y \end{matrix} \rightarrow \frac{dp^\mu}{d\tau} = q \begin{pmatrix} 0 \\ \gamma B \beta_y \\ \gamma B \beta_x \\ 0 \end{pmatrix}$$

$$P = -\frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp_m}{d\tau}\frac{dp^m}{d\tau}\right) = -\frac{2}{3}\frac{q^4}{m^2c^3}\left(-\gamma^2 B^2 \underbrace{\beta^2}_{\beta_x^2+\beta_y^2}\right) =$$

$$\boxed{P = \frac{2}{3}\frac{q^4 B^2 \gamma^2 \beta^2}{m^2 c^3}} = \frac{2 q^4 B^2 (\gamma^2 - 1)}{3 m^2 c^3}$$

b) find $E(t)$ if $\gamma \gg 1$

$P = -\frac{dE}{dt} \rightsquigarrow dE = -P\,dt \;;\; dE = d\gamma\, mc^2 = -P\,dt$

$$dt' = -\frac{mc^2\, d\gamma}{P} = \frac{3m^3c^5}{2q^4B^2}\,\frac{d\gamma}{\underbrace{(\gamma^2-1)}_{\sim\gamma^2}}$$

$$t = \int_0^t dt' = \frac{-3m^3c^5}{2q^4B^2}\int_0^{\gamma}\frac{d\gamma'}{\gamma'^2}$$

$$t = \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right) \longrightarrow E(t)$$

c)

$T = E - mc^2 = (\gamma-1)mc^2$

$$P = \frac{2q^4B^2}{3m^2c^3}(\gamma^2-1) = \frac{2q^4B^2}{3m^2c^3}\underbrace{(\gamma-1)}_{\sim T/mc^2}\overbrace{(\gamma+1)}^{\sim 2}$$

$$\boxed{P = \frac{4q^4B^2}{3m^3c^5}\,T} = -\frac{dE}{dt} = -\frac{dT}{dt} \Rightarrow \boxed{T = T_0 \exp\left(\frac{-4q^4B^2}{3m^3c^5}t\right)}$$

J# 14.21

4)

Virial theorem ~~⇒~~ $V = -2T$

~~E = T + V~~ and $\frac{mv^2}{r} = \frac{Ze^2}{r^2}$

for circular orbit:

$\Downarrow$

$$E = T + V = -\frac{Ze^2}{2r}$$

$$L = m\omega_0 r^2 = m\frac{v}{r}r^2 \xrightarrow[\text{Quantization}]{} m\omega_0 r^2 = n\hbar$$

$$\begin{cases} r_n = \dfrac{n^2\hbar^2}{Ze^2 m} \\ E_n = -\dfrac{Z^2 e^4 m}{2n^2\hbar^2} \end{cases}$$

Power radiated (classically) for n^{th} orbit

$$P_n = \frac{2}{3}\frac{Z^2 e^6 m^2 c}{n^8 \hbar^4}\left(\frac{Ze^2}{\hbar c}\right)^4$$

semiclassical transition from n to $n-1$

$$\frac{\hbar\omega_n}{\tau_n} = P_n \Rightarrow \frac{1}{\tau_n} = \frac{2}{3}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{\hbar c}\right)^4 \frac{mc^2}{\hbar} \frac{1}{n^5}$$

b)

semiclasical	quantum value	transition
3.0×10^{-9} s	1.6×10^{-9} s	$2p \to 1s$
9.6×10^{-8} s	7.3×10^{-8} s	$4f \to 3d$
7.3×10^{-7} s	6.1×10^{-7} s	$6h \to 5g$