Alf

1) a) (J 12.4)

$$\boxed{\mathcal{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha}$$

E.L. eqn: $$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = \frac{\partial \mathcal{L}}{\partial A^\nu} \qquad (\forall \nu)$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -\frac{1}{c} J_\alpha \frac{\partial A^\alpha}{\partial A^\nu} = -\frac{1}{c} J_\alpha \delta^\alpha_{\cdot\nu} = -\frac{1}{c} J_\nu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = -\frac{2}{8\pi} \partial_\alpha A_\beta \frac{\partial(\partial^\alpha A^\beta)}{\partial(\partial^\mu A^\nu)}$$

$$= -\frac{1}{4\pi} \partial_\alpha A_\beta \; \delta^\alpha_{\cdot\mu} \delta^\beta_{\cdot\nu}$$

$$= -\frac{1}{4\pi} \partial_\mu A_\nu$$

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} \right) = -\frac{1}{4\pi} \partial^\mu \partial_\mu A_\nu = -\frac{1}{4\pi} \Box A_\nu$$

$\Rightarrow$ E.L. eqn: $\boxed{\Box A_\nu = -\frac{4\pi}{c} J_\nu}$ Maxwell eqns only in Lorentz Gauge $(\partial_\mu A^\mu = 0)$

1) b)

$$\mathcal{L}' = -\frac{1}{6\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$$

$$\downarrow$$

$$\mathcal{L}' = -\frac{1}{6\pi}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) - \frac{1}{c} J_\alpha A^\alpha$$

$$= -\frac{1}{6\pi}\Big[\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha - \partial_\beta A_\alpha \partial^\alpha A^\beta - + \partial_\beta A_\alpha \partial^\beta A^\alpha\Big] - \frac{1}{c} J_\alpha A^\alpha$$

$$\mathcal{L}' = -\frac{1}{8\pi}\Big[\underline{\partial_\alpha A_\beta \partial^\alpha A^\beta} - \partial_\alpha A_\beta \partial^\beta A^\alpha\Big] - \frac{1}{c} J_\alpha A^\alpha$$

$$\mathcal{L}' - \mathcal{L} = \frac{1}{8\pi}\partial_\alpha\Big[\underbrace{A_\beta \partial^\beta A^\alpha}_{B^\alpha}\Big]$$

4-divergence of $A_\beta \partial^\beta A^\alpha$ (4 - vector

$$\Delta A = \int_V d^4x\,(\mathcal{L}' - \mathcal{L}) = \int_V d^4x\,\partial_\alpha B^\alpha = \int_S (B\cdot\hat{n})\,dS$$

↖ 4-dimensional surface.

surface terms cannot change equs. of motion.

1) c)

$$\mathcal{L} = k\bar{E}\cdot\bar{B} - \rho\phi + \frac{1}{c}\bar{J}\cdot\bar{A}$$

$$\mathcal{L} = k\left(-\bar{\nabla}\phi - \frac{1}{c}\frac{\partial\bar{A}}{\partial t}\right)\cdot\left(\bar{\nabla}\times\bar{A}\right) - \rho\phi - \frac{1}{c}\bar{J}\cdot\bar{A}$$

E.L. equ:
$$\begin{cases} \partial_t\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right) + \bar{\nabla}\cdot\dfrac{\partial\mathcal{L}}{\partial(\bar{\nabla}\phi)} = \dfrac{\partial\mathcal{L}}{\partial\phi} & (1) \\ \partial_t\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_t\bar{A})}\right) + \bar{\nabla}\times\left(\dfrac{\partial\mathcal{L}}{\partial(\bar{\nabla}\times\bar{A})}\right) = \dfrac{\partial\mathcal{L}}{\partial\bar{A}} & (2) \end{cases}$$

(trick) $\longrightarrow$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\rho \qquad \frac{\partial\mathcal{L}}{\partial\bar{\nabla}\phi} = -k\,\bar{\nabla}\times\bar{A} \qquad \bar{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial\bar{\nabla}\phi} = -k\,\bar{\nabla}\cdot(\bar{\nabla}\times\bar{A}) = 0$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = 0 \qquad \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right) = 0 \qquad \text{so } (1) \Rightarrow \boxed{\rho = 0}$$

$$\frac{\partial\mathcal{L}}{\partial\bar{A}} = -\frac{1}{c}\bar{J} \qquad \frac{\partial\mathcal{L}}{\partial(\bar{\nabla}\times\bar{A})} = k\left(-\bar{\nabla}\phi - \frac{1}{c}\frac{\partial\bar{A}}{\partial t}\right) \qquad \frac{\partial\mathcal{L}}{\partial(\partial_t\bar{A})} = -k\,\bar{\nabla}\times\bar{A}$$

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\bar{A})}\right) = -k\,\bar{\nabla}\times\partial_t\bar{A} \qquad \bar{\nabla}\times\left[\frac{\partial\mathcal{L}}{\partial(\bar{\nabla}\times\bar{A})}\right] = k\left(\underbrace{-\bar{\nabla}\times\bar{\nabla}\phi}_{\equiv 0} - \frac{\bar{\nabla}}{c}\times\frac{\partial A}{\partial t}\right)$$

$$\cancel{-\frac{k}{c}\,\bar{\nabla}\times\frac{\partial\bar{A}}{\partial t}} + \cancel{\frac{k}{c}\,\bar{\nabla}\times\frac{\partial\bar{A}}{\partial t}} = -\frac{1}{c}\bar{J} \quad \Rightarrow \boxed{\bar{J} = \bar{0}}$$

E.L. equ are not usefull at all.

2) (J 12.19)

a) $M^{\alpha\beta\gamma} = \Theta^{\alpha\beta} x^{\gamma} - \Theta^{\alpha\gamma} x^{\beta}$

$$\partial_\alpha M^{\alpha ij} = 0 \Rightarrow \int \partial_\alpha M^{\alpha ij} \, d^3x = 0$$

$$0 = \int \partial_0 M^{0ij} \, d^3x + \underbrace{\int \partial_k M^{kij} \, d^3x}_{\int_V \vec{\nabla} \cdot \left[\overrightarrow{M^{\bullet ij}}\right] d^3x = \cancel{\int_S \left[M^{\bullet ij}\right] \cdot d\vec{S}} \to 0}$$

$$0 = \partial_0 \int M^{0ij} \, d^3x$$

$$0 = \partial_0 \int \left[\Theta^{0i} x^j - \Theta^{0j} x^i\right] d^3x$$

$$0 = \partial_0 \int \left[\frac{1}{4\pi}\left(\bar{E} \times \bar{B}\right)_i x^j - \frac{1}{4\pi}\left(\bar{E} \times \bar{B}\right)_j x^i\right] d^3x \quad \forall ij$$

$$\Downarrow$$

$$\boxed{0 = \frac{d}{dt} \underbrace{\int \frac{\bar{x} \times \left(\bar{E} \times \bar{B}\right)}{4\pi c} \, d^3x}_{\vec{L}_{\text{fields}}}}$$

b) $\partial_\alpha M^{\alpha 0 \gamma} = 0 \quad \Rightarrow \quad \int \partial_\alpha M^{\alpha 0 \gamma} = 0$

$$0 = \int d^3x\, \partial_0 M^{00\gamma} - \int d^3x\, \partial_i M^{i0\gamma} \to 0$$

$$\boxed{0 = \int d^3x\, \partial_0 \left[\Theta^{00} x^\gamma - \Theta^{0\gamma} x^0 \right]}$$

$\underline{\gamma = 0}$ $\quad 0 = \int d^3x\, \partial_0 \left[\Theta^{00} x^0 - \Theta^{00} x^0 \right] \equiv 0$ (no info)

$\gamma \mapsto g = 1; 2; 3$

$$0 = \int d^3x\, \partial_0 \left[\Theta^{00} x^g - \Theta^{0g} x^0 \right]$$

$$0 = \frac{d}{dt}\left[c\, \frac{1}{8\pi} \int_V (E^2 + B^2) x^g \right] - \underbrace{\int d^3x\, \partial_0\, \Theta^{0g} x^0}$$

$$\int d^3x \left[\partial_0 \Theta^{0g}\right] x^0 + \Theta^{0g}$$

$$0 = \frac{d}{dt}\left[\int d^3x\, \frac{1}{8\pi} (E^2 + B^2) \frac{x^g}{c} \right] - \underbrace{\int dx^3\, \frac{\partial_0}{4\pi} (\bar{E} \times \bar{B})^g}\, x_0 - \int \frac{d^3x}{4\pi} (\bar{E} \times \bar{B})^g$$

$\frac{d}{dt} \bar{P}_{fields} = 0$ because $\bar{P}_{fields}$ is conserved

$$\frac{d}{dt}\int d^3x\, u\, \frac{\bar{x}}{c} = \frac{1}{4\pi}\int \underbrace{d^3x\,(\bar{E}\times\bar{B})}$$

$$\frac{d}{dt}\,\bar{\mathbb{X}}\int \frac{u\,d^3x}{c} = c\,\bar{P}_{field} \qquad c\,\bar{P}_{field}$$

$$\frac{1}{c}\left(\frac{d}{dt}\bar{\mathbb{X}}\right)\bar{E}_{fields} = c\,\bar{P}_{fields}$$

$$\frac{1}{c}\frac{d}{dt}\bar{\mathbb{X}} = \frac{c\,\bar{P}_{fields}}{\bar{E}_{fields}}$$

$$\boxed{\frac{d}{dt}\bar{\mathbb{X}} = c^2\,\frac{\bar{P}_{fields}}{\bar{E}_{fields}}}$$

3)
$$\begin{cases} \bar{\nabla}\cdot\bar{E} = 4\pi\rho \\ \bar{\nabla}\cdot\bar{B} = \emptyset \\ \bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \\ \bar{\nabla}\times\bar{B} = \frac{4\pi}{c}\bar{J}_\perp \end{cases}$$

$$\Rightarrow \exists \bar{A} / \boxed{\bar{B} = \bar{\nabla}\times\bar{A}}$$

$$\bar{\nabla}\times\left(\bar{\nabla}\times\bar{A}\right) = \frac{4\pi}{c}\bar{J}_\perp$$

$$\bar{\nabla}\left(\bar{\nabla}\cdot\bar{A}\right) - \nabla^2\bar{A} = \frac{4\pi}{c}\bar{J}_\perp$$

In <u>Coulomb Gauge</u> $\bar{\nabla}\cdot\bar{A} = \emptyset$

$$\boxed{\nabla^2\bar{A} = -\frac{4\pi}{c}\bar{J}_\perp}$$

$$\bar{A} = \frac{1}{c}\int_V \frac{\bar{J}_\perp(\bar{x}')}{|\bar{x}-\bar{x}'|}$$

$$\bar{\nabla}\times\bar{E} = \frac{-1}{c}\frac{\partial}{\partial t}\left(\bar{\nabla}\times\bar{A}\right)$$

$$\bar{\nabla}\times\left(\bar{E} + \frac{\partial}{\partial t}\frac{\bar{A}}{c}\right) = \bar{\emptyset} \Rightarrow \exists\phi / \bar{E} + \frac{\partial\bar{A}}{c\,\partial t} = -\bar{\nabla}\phi$$

so $\boxed{\bar{E} = -\bar{\nabla}\phi - \frac{1}{c}\frac{\partial\bar{A}}{\partial t}}$

$$\bar{\nabla}\cdot\left(-\bar{\nabla}\phi - \frac{1}{c}\frac{\partial\bar{A}}{\partial t}\right) = 4\pi\rho$$

$$\nabla^2\phi + \frac{1}{c}\frac{\partial(\bar{\nabla}\cdot\bar{A})}{\partial t} = -4\pi\rho + \text{Coulomb Gauge} \Rightarrow \boxed{\nabla^2\phi = -4\pi\rho}$$

$$\phi = \int \frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|}$$

Focus on 2 particle interaction:

$$\overset{1\leftrightarrow 2}{L_{int}} = -e\phi_2(\bar{x}_1) + \frac{e}{c}\,\bar{v}_1 \cdot \bar{A}_2(\bar{x}_1)$$

due to particle #2

$$\bar{J}_2(\bar{x}) = e\bar{v}_2\,\delta(\bar{x} - \bar{x}_2(t))$$

$$\Downarrow$$

$$\bar{J}_{2\perp}(\bar{x}) = e\,\bar{\bar{\Delta}}_\perp(\bar{x} - \bar{x}_2(t)) : \vec{v}_2 =$$

$$= e\bar{v}_2\,\delta(\bar{x} - \bar{x}_2(t)) - \frac{e}{4\pi}\bar{\nabla}\left[\frac{\bar{v}_2 \cdot (\bar{x} - \bar{x}_2(t))}{|\bar{x} - \bar{x}_2(t)|}\right]$$

$$\Rightarrow \bar{A}_2(\bar{x}_1) = \int \frac{\bar{J}_{2\perp}(\bar{x}')}{c|\bar{x}' - \bar{x}_1|}\,d^3x =$$

$$= \frac{e\bar{v}_2}{cR} - \frac{e}{4\pi c}\int \frac{1}{|\bar{x} - \bar{x}_2|}\bar{\nabla}\left[\frac{\bar{v}_2 \cdot (\bar{x}_1 - \bar{x}_2(t))}{|\bar{x}_1 - \bar{x}_2(t)|^3}\right] d^3\bar{x}$$

$$= \frac{e\bar{v}_2}{cR} - \frac{e}{2c}\bar{\nabla}_R\left(\frac{\bar{v}_2 \cdot \bar{R}}{R}\right) = \frac{e}{2c}\left[\frac{\bar{v}_2}{R} - \frac{\bar{R}(\bar{v}_2 \cdot \bar{R})}{R^3}\right]$$

where $\bar{R} = \bar{x}_1 - \bar{x}_2$

and $$\phi_2(\bar{x}_1) = e\int_V \frac{\rho_2(\bar{x}')}{|\bar{x}' - \bar{x}_1|}\, d^3x = \frac{e}{R}$$

so Lint becomes :

$$L_{int}^{1\leftrightarrow 2} = -\frac{e^2}{R} + \frac{e^2}{2c^2}\left[\frac{\bar{v}_1\cdot\bar{v}_2}{R} - \frac{(\bar{v}_1\cdot\bar{R})(\bar{v}_2\cdot\bar{R})}{R^3}\right]$$

by superposition principle (eqns are linear) this can be generalized to any number of particles.

Fields can be completelly eliminated and moreover interaction is instantaneous that is because this "pseudo Maxwell eqns" does not give any wave (finite velocity)* propagation of fields from sources.

$\equiv$ Darwin Lagrangian.

* Maxwell couldn't have predicted "light" (which velocity was already measured)