Physics 209 Homework I

Alfredo CORREA

3rd September 2002

1 Mean-value theorem

The mean value $\langle\phi\rangle$ of a potential ϕ in the surface of an sphere of radius R is:

$$\langle\phi\rangle(R) = \frac{R^2 \int d\psi \int \sin\theta d\theta \phi(R,\theta,\psi)}{R^2 4\pi}$$

Lets calculate explicitly the derivative $d\langle\phi\rangle/dR$, i.e. how the mean value varies when the radius of the sphere is changed:

$$\begin{aligned} \frac{d\langle\phi\rangle}{dR} &= \frac{1}{4\pi}\int d\psi \int \sin\theta d\theta \frac{\partial\phi}{\partial R}(R,\theta,\psi) \\ &= \frac{1}{4\pi}\int d\psi \int \sin\theta d\theta \nabla\phi\cdot\hat{\mathbf{r}} \\ &= \frac{1}{R^2 4\pi}\oiint_{\partial S} \nabla\phi\cdot d\mathbf{s} \end{aligned}$$

By Gauss theorem:

$$= \frac{1}{R^2 4\pi}\int\int\int_S \nabla^2\phi d^3\mathbf{r}$$

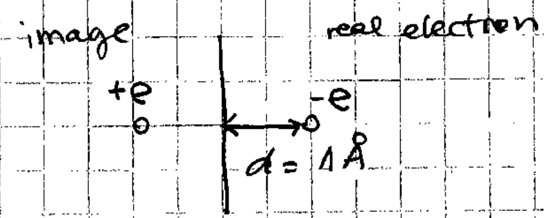
but $\nabla^2\phi = -4\pi\rho \equiv 0$ because by hypothesis there is no charges in the volume of the sphere, so:

$$\frac{d\langle\phi\rangle}{dR} = 0$$

and because of continuity $\lim_{R\to 0}\langle\phi\rangle(R) = \phi(0)$ the result is that:

$\langle\phi\rangle(R) = \phi(0)\ \forall R$ if no charges are included in the volume of the sphere .

(2.)



knowledge

$13\,eV \approx \frac{e^2}{2a_0}$

$a_0 \approx 0.5\,Å$

$d \approx 2 \times a_0$

$\text{Energy} = \frac{-e^2}{2 \times 2d}$

the work has an extra "2" because while bringing the electron from ∞ the image is "linked"

$\text{Energy} \approx \frac{-e^2}{2 \times 2 \times 2 \times a_0}$ → 13 eV

$\text{Energy} = \frac{-13\,eV}{4} \approx -3.25\,eV$

Work necessary to remove the electron a 1 Å from the surface

$\simeq +3.25\,eV$

(3)

a) By the image-charges method we find that: a given charge density in the semispace ($z \geq 0$) is "reflected" an changed in sign in to the other semispace ($z<0$) if we want to calculate the potential in the upper semispace (i.e. $z>0$); in mathematical form:

$$\phi_{z>0}(x;y;z) = \iiint_{z'>0} \frac{\rho(x';y';z')}{|\bar{r}-\bar{r}'|} d^3r' +$$

$$+ \iiint_{z'>0} \frac{(-)\rho(x';y';z')}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} d^3r'$$

but by definition

$$\phi_{z>0}(x;y,z) = \iiint \rho(x';y';z')\, G_D(\overbrace{x;y;z}^{\bar{r}};\overbrace{x';y';z'}^{\bar{r}'})\, dx'dy'dz'$$

comparing both expression we can choose this Green func.:

$$G_D(\bar{r};\bar{r}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{-1}{\sqrt{(x-x')^2+(y-y')+(z+z')^2}}$$

$\bar{r}=(x;y;z)$

$\bar{r}'=(x';y';z')$

$z>0 \quad z'>0$

③ b) Specifiing the potential over the surface is also a Dirichlet problem; the solution is:

$$\Phi(\bar{x}) = -\frac{1}{4\pi} \oint_{\partial S} \Phi(\bar{x}') \frac{\partial G_D}{\partial n'} \cdot da$$

in this case $\left.\frac{\partial G}{\partial n'}\right|_{\partial S} = \left.\frac{\partial G}{\partial z'}\right|_{z'=0}$

$$\frac{\partial G}{\partial z'} = \frac{(1/2) \cdot 1}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} (-)\, 2(z-z')$$

$$+ \frac{(-)(1/2) \cdot 1}{\left[(x-x')^2 + (y-y')^2 + (z+z')^2\right]^{3/2}} 2(z+z')$$

$$\left.\frac{\partial G}{\partial z'}\right|_{z'=0} = \frac{1}{\left[(x-x')^2 + (y-y')^2 + z^2\right]^{3/2}} \cdot \{\underbrace{-z - z}_{-2z}\}$$

$$\left.\frac{\partial G}{\partial z'}\right|_{z'=0} = \frac{-2z}{\left[(x-x')^2 + (y-y')^2 + z^2\right]^{3/2}}$$

$$\Phi(\bar{x}) = -\int_{-\infty}^{\infty} dx' \int_{\infty}^{\infty} dy' \, \Phi(x', y', 0) \frac{(-2z)}{\left[(x-x')^2 + (y-y')^2 + z^2\right]^{3/2}}$$

↓

(3 b) CONT

In cilindrical coordinates:

$$\Phi(\rho;\phi;z) = -\frac{1}{4\pi}\int_0^\infty \rho' d\rho' \int_0^{2\pi} d\phi' \underbrace{\Phi(\rho';\phi';0)}_{\text{known}} \frac{-2z}{\left[\rho^2 - 2\rho\rho'\cos(\phi'-\phi) + \rho'^2 + z^2\right]^{3/2}}$$

using that $\Phi(\rho';\phi';0) = V$ if $\rho' < a$
(0 otherwise)

$$\phi(\rho;\phi;z) = \frac{V}{4\pi}\int_0^a \rho' d\rho' \int_0^{2\pi} d\phi' \frac{2z}{\left[\rho^2 - 2\rho\rho'\cos(\phi'-\phi) + \rho'^2 + z^2\right]^{3/2}}$$

here we can choose $\phi = 0$ because the integral can not depend on ϕ

(3) c) On the axis

$$\phi(\rho=0; z) = \frac{V}{4\pi} \int_0^a \rho' d\rho' \int_0^{2\pi} d\phi' \frac{2z}{\left[\rho'^2 + z^2\right]^{3/2}}$$

$$= \frac{Vz}{4\pi} 2\pi \int_0^a \frac{2\rho' \, d\rho'}{\left[\rho'^2 + z^2\right]^{3/2}}$$

$$= Vz \frac{2\pi}{4\pi} \left. \frac{(-1)}{\sqrt{\rho'^2 + z^2}} \right|_{\rho'=0}^{\rho'=a}$$

$$= V(-z) \left\{ \frac{1}{\sqrt{a^2+z^2}} - \frac{1}{|z|} \right\}$$

as this valid for $z > 0 \Rightarrow |z| = z$

$$\phi(\rho=0; z) = V \left\{ 1 - \frac{z}{\sqrt{a^2 + z^2}} \right\}$$

in particular $\phi(\rho=0; z=0) = V$

③ a)

$$\left[\rho^2 - 2\rho\rho'\cos\phi' + \rho'^2 + z^2\right]^{-3/2} = (\rho^2+z^2)^{-3/2}\left\{1 + \lambda b\right\}^{-3/2}$$

where $\lambda = (\rho^2+z^2)^{-1}$

$b = -2\rho\rho'\cos\phi' + \rho'^2$

$$f(\lambda) = \left\{1 - \lambda b\right\}^{-3/2}$$

$$f'(\lambda) = \frac{3}{2}\left\{1 - \lambda b\right\}^{-5/2}(+b)$$

$$f''(\lambda) = +\frac{3}{2}\left(+\frac{5}{2}\right)\left\{1 - \lambda b\right\}^{-7/2} b^2$$

$$f^{(N)}(\lambda) = \frac{(2N+1)!!}{2^N}\left\{1-\lambda b\right\}^{\frac{-3+2N}{2}} b^N$$

$\vdots$

$$f(\lambda) \approx f(0) + f'(0)\lambda + f''(0)\lambda^2/2 + \cdots$$

$$\approx 1 + \frac{3}{2} b\lambda + \frac{15}{4} b^2 \frac{\lambda^2}{2} + \cdots$$

$\Downarrow$

$$\left[\rho^2 - 2\rho\rho'\cos\phi' + \rho'^2 + z^2\right]^{-3/2} = (\rho^2+z^2)^{-3/2}\Big[1 + \frac{3}{2}\left(-2\rho\rho'\cos\phi' + \rho'^2\right)\lambda +$$

$$+ \frac{15}{4}\left(-2\rho\rho'\cos\phi' + \rho'^2\right)^2\lambda^2\Big]$$

③d CONT.

$$\phi(\rho;\phi;z) = \frac{Vz}{2\pi}\lambda^{3/2} \int_0^a\int_0^{2\pi} d\phi' \rho' d\rho' \left[1 - \frac{3}{2}\left\{-2\rho\rho'\cos\phi' + \rho'^2\right\}\lambda + \frac{15}{8}\left\{-2\rho\rho'\cos\phi' + \rho'^2\right\}^2 \lambda^2 \right]$$

the linear terms of $\cos\phi'$ integrates to zero because $\int_0^{2\pi} d\phi' \cos\phi' = 0$

while the quadratic terms integrates to π because $\int_0^{2\pi} d\varphi' \cos^2\varphi = \frac{2\pi}{2}$

Thus:

$$\phi(\rho;\phi;z) = \frac{Vz\lambda^{3/2}}{2\pi} \int_0^a \rho' d\rho' \left[2\pi - \frac{3}{2} 2\pi \rho'^2 \lambda + \left(\frac{15}{8} \cdot 4\rho^2\rho'^2\pi + \frac{15}{8}\rho'^4 2\pi\right)\lambda^2 \right]$$

$$= \frac{Vz\lambda^{3/2}}{2\pi} \left[\frac{a^2}{2} 2\pi - \frac{3}{2}2\pi \frac{a^4}{4}\lambda + \left(\frac{15}{8} 4\rho^2 \frac{a^4}{4}\pi + \frac{15}{8}\frac{a^6}{6} 2\pi \right)\lambda^2 \right]$$

$$= \frac{Vz\lambda^{3/2}}{2} a^2 \left[1 - \frac{3}{4}a^2\lambda + \frac{15}{8}\left(\rho^2 a^2 + a^4/3\right)\lambda^2 + \dots \right]$$

$$\phi(\rho,\phi,z) \cong \frac{Va^2}{2} \frac{z}{(\rho^2+z^2)^{3/2}} \left[1 - \frac{3}{4}\frac{a^2}{(\rho^2+z^2)} + \frac{5}{8}\frac{(3\rho^2a^2 + a^4)}{(\rho^2+z^2)^2} + \dots \right]$$

if ~~$\rho = 0 \Rightarrow \phi(\rho=0,\phi,z) = \frac{Va^2}{2}\frac{z}{|z|^3}$~~

(3) d) ...

Comparison between part d) and part c)

The result of c) is

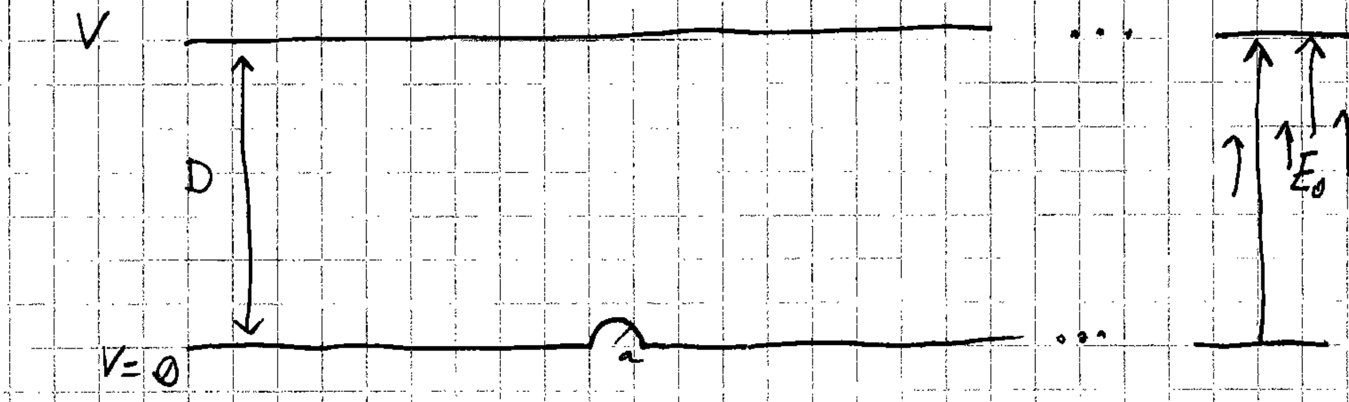
$$\phi(\rho=0;z) = V\left\{1-\frac{z}{\sqrt{a^2+z^2}}\right\} = V\left\{1-\frac{1}{\sqrt{1+\frac{a^2}{z^2}}}\right\}$$

$$\approx V\left\{1-1+\frac{1}{2}\frac{a^2}{z^2}+\ldots\right\} = V\left\{\frac{1}{2}\frac{a^2}{z^2}\right\}+\ldots$$

↑ for large z i.e. $a^2 \ll z^2$

↑ and this is the first term of the result of part d) (previous page)

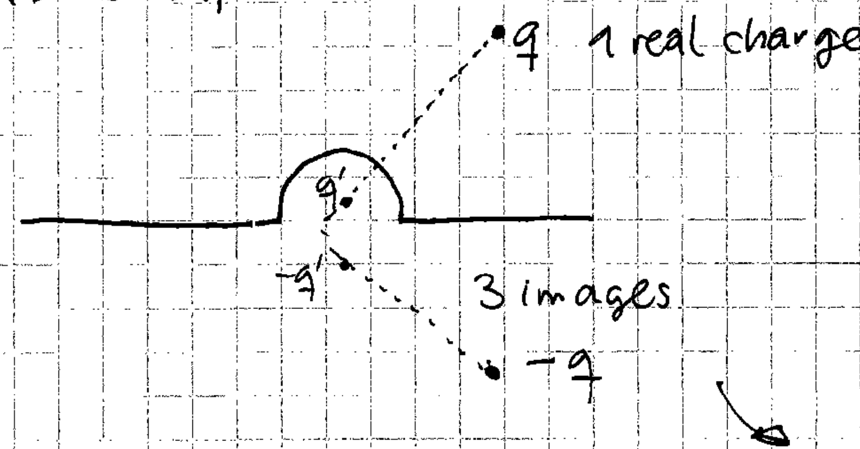
(4)

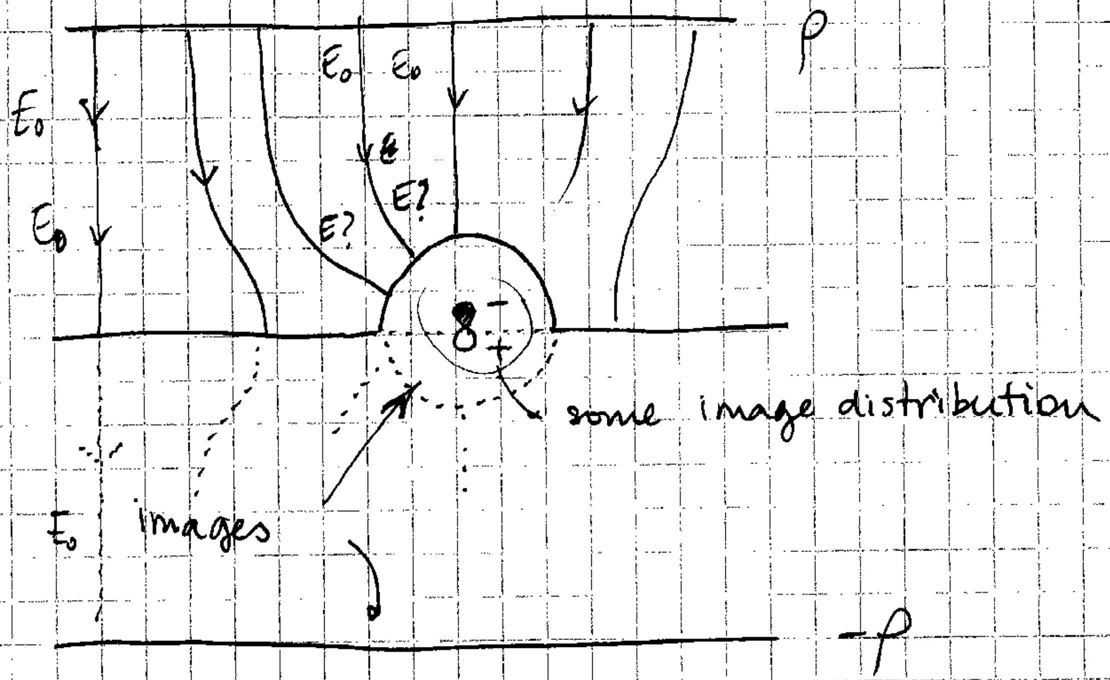


$V = -E_0 D$

a) If $a \ll D$ the charge distribution in the upper plane is not affected by the presence of the boss; i.e. remains uniform $\rho_{upper\,plate}(x;y) \equiv \rho_0 = \frac{E_0}{4\pi}$

Now this charge distribution has its image in the other plate (with the boss) as any charge as its image in this configuration. Example:





.. But if $D \gg a$ this is the problem of the conducting sphere inserted in an homogeneous (at ∞) field $E_0 \hat{z}$

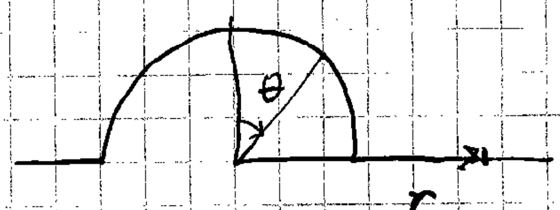
And we know the solution:

$$\begin{cases} \bar{E} \cdot \hat{r} = E_0 \left(1 + 2\dfrac{a^3}{r^3}\right)\cos\theta \\ \bar{E} \cdot \hat{\theta} = -E_0 \left(1 - \dfrac{a^3}{r^3}\right)\sin\theta \end{cases}$$

and $\sigma = \epsilon_0 \Delta E_\perp$

⇓ (on the boss)

$$\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta$$



$$\sigma(r) = -\epsilon_0 E_0 \left(1 - \frac{a^3}{r^3}\right)$$

(on the plane)

(4) b) The total charge on the boss is then:

$$Q_{boss} = \int_0^{2\pi}\int_0^{\pi} \sigma(\theta)\, a^2 \sin\theta \, d\theta \, d\varphi$$

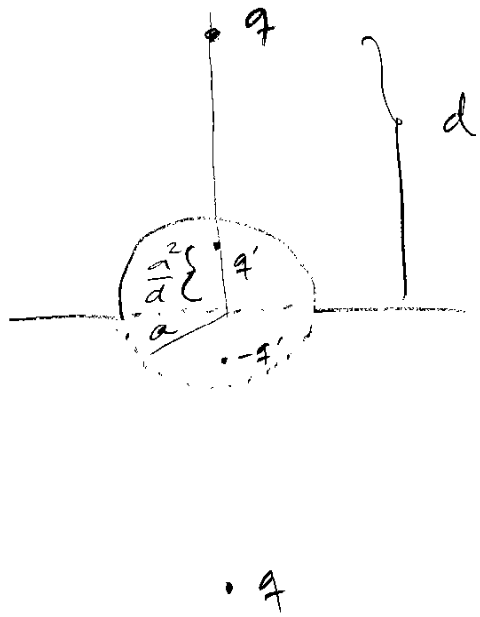
$$Q_{boss} = -2\pi a^2 \int_0^{\pi} 3 E_0 \epsilon_0 \cos\theta \sin\theta \, d\theta$$

$$Q_{boss} = -2\pi a^2 \, 3 E_0 \epsilon_0 \underbrace{\int_0^{\pi} \cos\theta \sin\theta \, d\theta}_{0}$$

$$\underbrace{-\frac{1}{2}\cos^2\theta \Big|_0^{\pi}}_{-\frac{1}{2}(\not{0} - 1)}$$

$$\boxed{Q_{boss} = -3\pi a^2 \epsilon_0 E_0}$$

4) c) This is a problem equivalent to that of 4 charges



The total potential is given by Jackson:

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r^2+(z-d)^2)^{1/2}} - \frac{1}{\sqrt{r^2+(z+d)^2}} - \frac{\frac{a}{d}}{\sqrt{r^2+\left(z-\frac{a^2}{d}\right)^2}} + \frac{\frac{a}{d}}{\sqrt{r^2+\left(z+\frac{a^2}{d}\right)^2}}\right]$$

The charge in the plane is $\sigma_{plane} = -\epsilon_0 \left.\frac{\partial\phi}{\partial z}\right|_{z=0}$

$$\sigma_{plane} = -\frac{1}{2\pi} q \left[\frac{d}{(r^2+d^2)^{3/2}} - \frac{a^3}{d^2}\frac{1}{\left(r^2+\frac{a^4}{d^2}\right)^{3/2}}\right]$$

So the total charge in the plane is

$$Q_{plane} = \int_a^\infty \int dr\, d\phi\, r\ \sigma_{plane}(r)\, d\phi\, dr$$

$$= -\frac{q}{2\pi} \int d\phi \int_a^\infty \left[\frac{d}{(r^2+d^2)^{3/2}} - \frac{a^3}{d^2} \frac{1}{\left(r^2+\frac{a^4}{d^2}\right)^{3/2}} \right] r\, dr$$

$$= +q \left(\frac{a^2-d^2-}{d\sqrt{a^2+d^2}} \right)$$

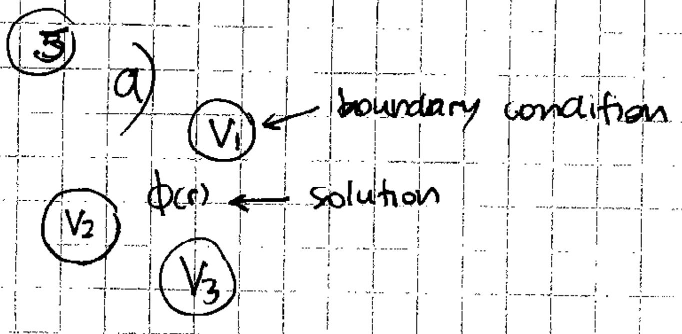
By Gauss law: we know that

$$q + Q_{plane} + Q_{boss} = 1$$

so

$$\boxed{Q_{boss} = -q\left[1 - \frac{d^2-a^2}{d\sqrt{a^2+d^2}}\right]}$$

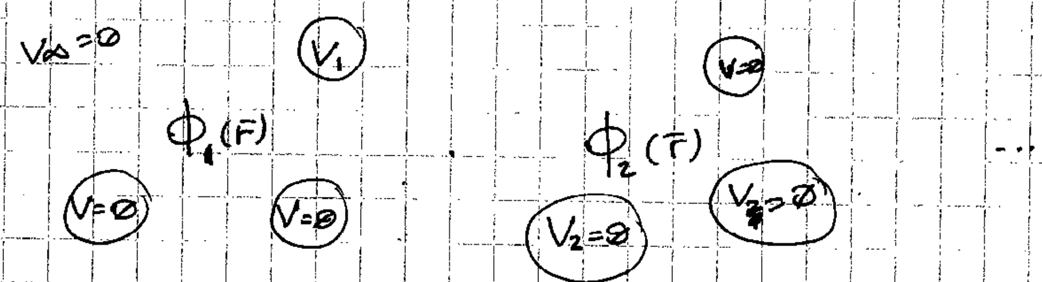
(3) a)



by linearity

$$\phi(\bar{r}) = \phi_1(\bar{r}) + \phi_2(\bar{r}) + \dots$$

where $\phi_i(\bar{r})$ is the solution of the problems:



Moreover $\phi_1(\bar{r}) \propto V_1 \Rightarrow \phi_1(\bar{r}) = \alpha_1(\bar{r}) V_1$

$\phi_2(\bar{r}) \propto V_2 \Rightarrow \phi_2(\bar{r}) = \alpha_2(\bar{r}) V_2$

Then:

$$\phi(\bar{r}) = \alpha_1(\bar{r}) V_1 + \alpha_2(\bar{r}) V_2 + \dots$$

where $\alpha_i(\bar{r})$ only ~~..~~depend on the geometry

the charge density $\sigma(\bar{r})$ over the surface is given by $\sigma(\bar{r}) = \bar{E} \cdot \hat{n}$ over the surfaces.

↘

and the total charges are

$$Q_1 = \int_{S_1} \sigma(\bar{r})\, ds = \int_{S_1} \bar{E}(\bar{r}) \cdot d\bar{s}$$

$$= \int_{S_1} -\bar{\nabla}\phi(\bar{r})\, d\bar{s} = -\int_{S_1} \left[\bar{\nabla}\phi_1(\bar{r}) + \bar{\nabla}\phi_2(\bar{r}) \cdots\right] \cdot d\bar{s}$$

$$= -\int_{S_1} \left[\bar{\nabla}\alpha_1(\bar{r}) V_1 + \bar{\nabla}\alpha_2(\bar{r}) V_2 + \cdots\right] \cdot d\bar{s}$$

$$Q_1 = \left[-\int_{S_1} \bar{\nabla}\alpha_1(\bar{r})\right] V_1 + \left[-\int_{S_1} \bar{\nabla}\alpha_2(\bar{r})\right] V_2 + \cdots$$

and the same for each conductor:

$$Q_2 = \left[-\int_{S_2} \bar{\nabla}\alpha_1(\bar{r})\right] V_1 + \left[-\int_{S_2} \bar{\nabla}\alpha_2(\bar{r})\right] V_2 + \cdots$$

$\vdots$

That means that exist a matrix such that:

$$\boxed{Q_i = \sum_j C_{ij} V_j}$$

i.e. exist a linear relation between $\{\bar{Q}\}$ and $\{\bar{V}\}$

where $\boxed{C_{ij} = -\int_{S_i} \bar{\nabla}\alpha_j(\bar{r})\, ds}$ and α satifies $\boxed{\alpha_i\big|_{S_j} = \delta_{ij}}$ and $\boxed{\nabla^2\alpha_i = 0}$

Q.E.D.

(5) b)

For ~~a particular~~ any choise of $\{V_i\}$ there is an associated field $\vec{E} = -\vec{\nabla}\phi$ where ϕ is the unique solution to

$$\nabla^2 \phi = 0 \quad \text{and} \quad \phi\Big|_{S_i} = V_i$$

The energy of the system is

$$\frac{\epsilon_0}{2} \int_V |E|^2 d^3r > 0$$

or

$$\frac{1}{2} \int \rho(x)\, \phi(x) = \frac{1}{2} \sum_{i} V_i Q_i = \frac{1}{2} \sum_{ij} C_{ij} V_i V_j$$

Then

$\sum_{ij} C_{ij} V_i V_j > 0$ (for any $\{V\}$) and then the matrix $\{C_{ij}\}$ is positive definite

⑤ c)

$$\begin{cases} Q_1 = C_{11} V_1 + C_{12} V_2 \\ Q_2 = C_{21} V_1 + C_{22} V_2 \end{cases}$$

The usual case is that of two conductors with opposite charges ($Q_1 = -Q_2 = Q$) and C is defined as $\frac{Q}{\Delta V}$.

$$\begin{cases} Q = C_{11} V_1 + C_{12} V_2 \\ -Q = C_{21} V_1 + C_{22} V_2 \end{cases} \Rightarrow \quad V_1 = \frac{Q C_{22} + Q C_{21}}{C_{11} C_{22} - C_{12} C_{21}}$$

$$V_2 = \frac{-Q C_{11} - Q C_{21}}{C_{11} C_{22} - C_{12} C_{21}}$$

$$V_1 - V_2 = Q \frac{C_{22} + C_{21} + C_{21} + C_{11}}{C_{11} C_{22} - C_{12} C_{21}}$$

$$C = \frac{C_{11} C_{22} - C_{12} C_{21}}{C_{22} + C_{21} + C_{12} + C_{11}} > 0$$

$C_{11} C_{22} - C_{12} C_{21} > 0$ (prop. of any positive definite matrix) (the determinant is positive)

$C_{22} + C_{21} + C_{12} + C_{11}$ also > 0 (it is other property of any definite matrix)

$\boxed{C > 0}$ Q.E.D.

demo:

$$(1\ 1) \cdot \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = C_{11} + C_{22} + C_{21} + C_{12} > 0$$

because it is positive definite

(6) To the effect of calculating the potential at the central point of a regular poliedron there is an equivalence between many problems: