Lattice basis vectors $\hat{e}_k \Rightarrow$ Basis contours $\Gamma_k$.

Review gate p.2.

Just since we found that the phase space evolutions generated by the constants of motion in an integrable system can be parametrized by coordinates in a vector space which we represent here by $\mathbb{t}$-space.

We also found that the surface swept out by these various evolutions from some fixed initial condi is a torus.

We then obtain an association between basis vectors in $\mathbb{t}$-space with basis contours on the torus.

Then the fundamental parallelogram in $\mathbb{t}$-space corresponds in a 1-1 manner with the torus itself, with the understanding that opposite sides of the parallelogram represent the same points in phase space.
As you can see, the period vectors $E_1, E_2$ in $T$-space correspond to basic contours on the torus. Call them $\Gamma_1, \Gamma_2$.

Make a definition of basic contours:

**Definition:** A set of closed contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ on an $N$-torus forms a basis if every closed contour $C$ can be written

$$C = \sum_{i=1}^{N} n_i \Gamma_i$$

where $n_i$ are integers.

Example:
- Draw a closed curve, goes once around long way, once around short way
- $C = \Gamma_1 + \Gamma_2$

When you draw a torus imbedded in 3D space like this, there's a misleading feature that you should be wary about.

Namely, on such an imbedded torus, you have a privileged choice for basic contours, namely the "long way" and "short way".
- But this is not intrinsic to the torus itself, only the imbedding.
- But there is no such privileged choice on a torus in phase space; any basis is as good as any other basis.

So this leads us into the subject of a change of basis.

Write

$$\Gamma'_i = \sum_{j=1}^{N} B_{ij} \Gamma_j$$

change of basis.
Now, if this were linear algebra, you would demand that \( \det B \neq 0 \), so an old basis would go into a new one.

But here we have contours, not vectors, and therefore \( \Gamma' \) must also be closed contours on the torus, i.e., components of \( \Gamma' \) must lie integers. So conclude, \( B = \text{nonsingular integer matrix} \).

But that's not all. Can invert, get

\[
\Gamma_i = \sum_{j=1}^{N} (B^{-1})_{ij} \Gamma' \tag{1}
\]

And by same argument, \( B^{-1} \) must also be an integer matrix.

Now consider eqn., \( (\det B)(\det B^{-1}) = 1 \).

Both must be integers; product of 2 integers \( = 1 \) \( \Rightarrow \) each is \( \pm 1 \).

\[
\begin{align*}
\text{Change of basis contours,} \\
\Gamma' &= B \cdot \Gamma, \\
\text{N x N} \\
B &= \text{integer matrix, } \det B = \pm 1.
\end{align*}
\]

Go back to our \( F \) space parallelograms,

\[
\begin{pmatrix}
\Gamma_1' \\
\Gamma_2'
\end{pmatrix} = \begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\Gamma_1 \\
\Gamma_2
\end{pmatrix}
\]

You see, the fundamental period parallelogram is not unique, either.
Now let's consider the question of coordinate systems on phase space. In the previous lecture, we introduced the concepts of phase space and integrable systems. In an integrable system, we have a set of constants of motion. As we know, it's natural to try to use these constants as part of a coordinate system on phase space. But why is the set of variables we need? Where are we going to get the other half?

Answer is obvious, just use the \( \ell \) variables:

\[
(\vec{q}, \vec{p}) \rightarrow (\vec{\ell}, \vec{A}).
\]

Notice the geometrical significance: The \( A_i \)'s identify a certain simultaneous contour surface, which for bound systems we recognize as tori, and which for any system are surfaces of the \( \ell_i \)'s, and the \( \ell \)'s label points on these surfaces.

Now this leads us to the subject of action-angle variables.

The idea of these is simple: it is simply that the \( \ell \)-coordinate is kind of ugly, because the periodicities are not simple. 2 problems:

Not only are the basis vectors not aligned with the axes, but also the lengths of these vectors, i.e., the measure of the periods, are not something simply like \( 2\pi \), but are \( \text{fin} \) of the \( A \)'s.

On the other hand, it's natural to use purely angle variables on a torus, call them \( \bar{\ell} \), which vary between \((0, 2\pi)\) on going around some choice of basis contours.

Can relate to the \( \ell \)-coordinates by

\[
\ell(\bar{\ell}) = \ell = \sum_{k=1}^{N} e_k \frac{\ell_k}{\partial \ell_k},
\]
Let's analyze this question first in 1D.

Now let's look for a canonical transformation,

$$(x, p) \rightarrow (\Theta, I),$$

where $\Theta$ will be some "nice" angle coordinate on the torus, and $I$ will be canonically conjugate to $\Theta$. How do we find this?

Well, first solve it in 1D.

$$A_1 = H.$$

Only 1 t-coord, and that is time. So, we want

$$(x, p) \rightarrow (\Theta, I).$$

Now it's natural to look for a transformation of which $I = s(H)$, or, in this case, $K = s(H)$. Reason is, we want $s$ to label the tori as well as $I$; we also want our new coordinates to respect the foliation, as well as $(t, H)$. So demand:

$$\Theta = 2\pi \cdot \text{periodic},$$

$$I = s(H).$$

The way to find this $s$ is to use a gen. $F$. We'll use the $F_2$ gen. $F$

$$S = S(x, I) = F_2(x, I).$$

Then,

$$\Theta = \frac{\partial S}{\partial I}(x, I), \quad p = \frac{\partial S}{\partial x}(x, I).$$
The 2nd eqn. here allows us to find $S$. It gives,

$$ S(x, \pi) = \int_{x_0}^{x} p \, dx $$

But geometrically, you can see this in phase space as an integral along the orbit.

An interesting feature of this is that $S$ is actually a multiple valued function of $x$.

$$ \Delta S = \int_{\text{const.}} p \, dx = A = A(I) = \text{area} $$

Define action $I = \frac{1}{2\pi} \int_{\text{orbit}} p \, dx$.

Note, $I = I(H) = I(f)$. Notice, this multi-valuedness of $S$ doesn't affect the $p$ eqn., because $\Delta S$ is indep. of $x$. But it does affect the $\theta$ eqn.

$$ \Delta \theta = \frac{\partial}{\partial f} \Delta S = 2\pi \frac{2I}{\partial f} $$

But this is just what we want; demand $\Delta \theta = 2\pi$, and we have

$$ \frac{2I}{\partial f} = 1 $$

Therefore we conclude that the variable conjugate to our desired angle is none other than the action.
So now that we recognize that we don't need $I$ any more, just write,

$$(x, p) \rightarrow (\theta, I), \quad I = I(H) = \frac{1}{2\pi} \oint_{\theta = \text{const}} p \, dx.$$ 

Notice what happens when we transform $H$ to the new coords. By construction, $H$ is a fn. only of $I$, inverse of fn. above, so

$$H = H(I), \quad \dot{\theta} = \frac{\partial H}{\partial I} = \omega(I),$$
$$\dot{I} = -\frac{\partial H}{\partial \theta} = 0.$$ 

Now let's go back to multidimensional case. Let's just look for a CT. Now that we recognize that actions are important,

$$(x, p) \rightarrow (\theta, I)$$

where we define the actions by

$$I_k = I_k (R) = \frac{1}{2\pi} \oint_{R} \vec{F} \cdot d\vec{x}.$$ 

To find out what the variables conjugate to $\dot{\theta}$ are, we set

$$S(x, \theta) = \int_{\theta}^{\theta + \Delta \theta} F \cdot d\theta.$$ 

As in the 1D case, this integral is multiply periodic. But now there are many periods, so you can see from the picture,

$$\Delta k S = 2\pi I_{k}.$$
Now apply the gen. fn. relations,

\[ \tilde{p} = \frac{\partial S}{\partial \tilde{\epsilon}} (\tilde{\phi}, \tilde{\xi}) \quad \Delta \tilde{p} = 0 \]

\[ \tilde{\theta} = \frac{\partial S}{\partial \tilde{\phi}} (\tilde{\phi}, \tilde{\xi}) \quad \Delta \tilde{\phi} = 2\pi \]

Finally, let's consider the transformation of the Hamiltonian. First observe that it must be independent of \( \tilde{\phi} \), since the tori are invariant manifolds, and therefore are all at fixed energy. Therefore

\[ H = H(\tilde{\xi}) \]

Then, Ham's equs are,

\[ \dot{\tilde{\phi}} = \frac{\partial H}{\partial \tilde{\xi}}, \quad \dot{\tilde{\xi}} = \frac{\partial H}{\partial \tilde{\phi}} \]

\[ \frac{\partial}{\partial \tilde{\phi}} = 0 \]

Notice also, that since the \( \theta \)'s are all \( 2\pi \)-periodic on each torus, that \( \tilde{\xi}, \tilde{p} \) are now periodic fun of \( \theta \):

\[ \tilde{\xi} = \tilde{\xi}(\theta, \tilde{\xi}) \] multiply periodic in \( \theta \)

\[ \tilde{p} = \tilde{p}(\theta, \tilde{\xi}) \]

Can expand in Fourier series,

\[ \tilde{\xi}(\theta, \tilde{\xi}) = \sum_{n} \tilde{\xi}_{n}(\tilde{\xi}) e^{i \pi \cdot \theta} \quad \tilde{\xi} = (n_1, \ldots, n_N) \]

etc.
Don't try to justify actions anymore, just use them in multi-dim.

Definition of actions, $N > 1$ D.A.E.

$$I_{A} = I_{A}(\mathbf{A}) = \frac{1}{2\pi} \oint_{\mathbf{A}} \mathbf{A} \cdot d\mathbf{\ell}$$

Notes:

1. $\oint \mathbf{A} \cdot d\mathbf{\ell}$ path integral
2. Depends on basis contours — $I_{A}$ is subject to some arbitrariness as $I_{A}$
3. $I_{A}$ is a function of $\mathbf{A}$'s, and conversely. Use $I_{A}$ to parameterize tori.
Given \((x, \vec{p}) \rightarrow (\tilde{x}, \tilde{\vec{p}})\), prove \(\tilde{\varphi} = \tilde{\varphi}_0 \cdot \tilde{\Theta} \).

1. \(H = H(\tilde{x})\)  \(\Rightarrow\) \(\{\text{If } H = A_k, \text{ then }\)

\[\tilde{H} = \tilde{A}_k(\tilde{x})\]

\(\tilde{\varphi}_0 = \tilde{\varphi}_0(\tilde{x})\).

2. Equations of motion:

\[\tilde{\varphi} = \frac{\partial H}{\partial \tilde{\vec{p}}}_k = \varphi_k(\tilde{x}).\]

\[I_k = 0.\]

\[\tilde{\varphi} = \tilde{\varphi}(\tilde{x}) + \tilde{\varphi}_0(\tilde{x})t.\]

End of 3/4/86: 22 down

21 to \(\varphi_0\).