Last time we saw that the HT eqns only have solutions when smooth, invariant LM's exist, and we also saw that sometimes they don't. (Namely when you have chaotic orbits.)

Well, there is a certain class of systems for which such smooth inv. LM's always exist, and these are the integrable systems.

So today we will begin a discussion of integrable systems, and take them up to the point of finding energy levels and wave func.

Begin with some defs. A lot of defs and these today, most of them easy.

Def: Two functions \( F(\vec{x},\vec{p}) \) and \( G(\vec{x},\vec{p}) \) are said to be in involution if

\[
\{ F, G \} = 0.
\]

(\text{curly } \{ \text{ is PB}. \text{ Obviously equivalent of commuting observable} \) in G.M.

Def: A time-indep. system of N DOF is integrable if there exist N independent constants of motion, \( A_1(\vec{x},\vec{p}), \ldots, A_N(\vec{x},\vec{p}), \) in involution.

Notice, since the fun's \( A_k \) are cons of motion, they all commute with \( H: \)

\[
\{ A_k, H \} = 0 \quad \{ A_k, A_l \} = 0.
\]

Notice also, if you have some system and you're looking for cons of motion in involution, hoping to find enough, you can always start with \( H \) itself.
you can see, an immediate corollary:

**Corollary:** Every system of 1 D.O.F. is integrable.

Take \( A_1 = H \).

Another simple corollary:

**Corollary:** Every system of 2 D.O.F. with some constant \( F \) indep. of \( H \) is integrable.

\[
A_1 = H, \quad A_2 = F, \quad \varepsilon A_1, A_2^2 = -i \varepsilon F, H^2 = 0.
\]

Now, it's important in this defn. that the constants be in involution. For example, take

\[
H = \frac{\dot{r}^2}{2m} + V(r) \quad (3 \text{D.O.F.})
\]

Can write down lots of constants:

\[
H, L_x, L_y, L_z. \quad 4 \text{ cons., all indep.}
\]

But not in involution. How many in inv. ?

2 from list given \((H, L_z)\)

but 3 by forming freem \((H, L_z, L^2)\).

So, this system is integrable, as are all rotationally symmetric systems of 3 D.O.F.
Comment: The constants in involution are obviously the analog of a complete set of commuting observables in CM.

Another comment concerns this word independent, which occurs in the defn. What does this mean? Obviously, $Lx, Lz$ not indep. Makes sense: if it is a const, you don't want to count $N^2$ as another const. Also, you want to exclude consts that are really const.

Const in defn $\Rightarrow$ const along orbit

Exclude: consts everywhere in phase space.

Otherwise, $A_1(\gamma, \beta) = 1$

$$A_2 = 2$$

eht.

and every system would be integrable.

Here's the defn:

Defn: A set of funs $A_1(\gamma), \ldots, A_k(\gamma)$ are independent (at a point, over a region, eht) if $\nabla A_1, \ldots, \nabla A_k$ are linearly indep.

Note: excludes absolute constants; $A = 3 \Rightarrow \nabla A = 0$.

Note also: $\nabla^2 \Psi = 2\Psi \nabla \Psi$.

Sometimes you can have constants of motion which are independent almost everywhere, but which degenerate into dependency at a subset of phase space (usually a surface of lower dimensionality). This can happen even in 1DOF. Here only const. you need is $H$.  


We want $\nabla H \neq 0$, $H$ indep. const.
$\nabla H = 0$, $H$ not indep.

Well, if $\nabla H = 0$, then $\dot{z} = \nabla H = 0$, and we have a fixed point.

You see, the orbits going into the unstable fixed points are the separatrices, and these in a general sense are associated with the breakdown of the independence of the const. of motion.

Now, what are cons. of motion good for? Let's look at this in terms of the theory of ODE's, without regard to Hamiltonian mechanics.

Write down any old system of ODE's,

$$
\dot{x}_1 = F_1(x_1, \ldots, x_n) \\
\vdots \\
\dot{x}_n = F_n(x_1, \ldots, x_n).
$$

Now a cons. of motion is a for

$$
\alpha(x_1, \ldots, x_n) \quad \text{const. along orbits, i.e.}
$$

$$
\sum_{i=1}^{n} \frac{\partial \alpha}{\partial x_i} \dot{x}_i = 0.
$$
Now what good this does is it allows you to eliminate one variable from the system of ODE's, e.g. solve \( a(x_1, \ldots, x_n) = c \) for \( x_n \), eliminate \( x_n \).

So, in the general case, the knowledge of each constant allows you to reduce your system by one variable.

Therefore if we had a Ham. syst. with \( N \) consts., you would naively think that you could reduce the \( 2N \) coupled ODE's to a new set of only \( N \) coupled ODE's.

But if the constants are in involution, then it turns out you can solve the whole thing. This is known as

\textbf{Liouville's Thm.:} If a system is integrable, it can be solved by quadratures.

In other words, constants in involution are twice as powerful as just any old constants.

Not the same as usual Liouville's Thm., also called Krull's Thm.

It will take us some time to prove Liouville's Thm. Begin with a more modest thing, that will take us there.

Consider an integrable system,

\[ A_1, \ldots, A_N = \text{consts. in involution}. \]

Consider \( \mathcal{M} = \text{surface} \ A_1(\vec{x}, \vec{p}) = c_1 \quad \implies \quad (N\text{-dim.}) \quad A_N(\vec{x}, \vec{p}) = c_N \)
Then, if \( A_1, \ldots, A_n \) are independent on \( M \), then \( M \) is an invariant LM.

(over)

Proof of this is easy. \( A_1, \ldots, A_n \) are independent \( \Rightarrow \) \( \nabla A_1, \ldots, \nabla A_n \) are linearly independent.

The \( \nabla \)'s are linearly independent because \( J \) is nonsingular.

Set \( X_k = J \cdot \nabla A_k \) now it's easy to show that the \( X_k \)'s are skew-orthogonal:

\[
\nabla A_k \cdot \nabla A_l = \nabla A_k \cdot J \cdot J^{-1} \cdot \nabla A_l = \nabla A_k \cdot J \cdot \nabla A_l
\]

\[
= \{ A_k, A_l \} = 0.
\]

This doesn't yet show that \( M \) is an LM, because we haven't shown that the \( X_k \)'s are tangent to \( M \). To do that, we need to show that all of the \( A_k \)'s are constants along \( X_k \):

\[
\nabla A_k \cdot \nabla A_k = \nabla A_k \cdot J \cdot \nabla A_k = \{ A_k, A_k \} = 0.
\]

This was for a single \( M \), single surface, but instead of count's \( \xi_1, \ldots, \xi_n \),

**Cor:** If \( A_1, \ldots, A_n \) are independent everywhere, they foliate phase space into an \( N \)-param. family of Lagrangian manifolds.

Now let's go back to our vector fields \( X_k \), and explore their properties. Notice that \( J \cdot \nabla A_k \) looks just like Ham's eqn, if \( A_k \) is treated as a Hamiltonian. That is, we might interpret \( X_k \) as a velocity vector, and etc

\[
X_k = \frac{dz}{dt_k} = J \cdot \nabla A_k
\]

(Of course, unless \( A_k = H, + \) is not time, just some param.

Since many \( A_k \)'s, each with own param. call it \( k \).
$A_{\alpha}, \ldots, A_{N} = \text{indep.}$

$⇒ \nabla A_{\alpha}, \nabla A_{\beta} = \text{lin. indep.}$

let $X_{\alpha} = J \nabla A_{\alpha}$.

$⇒ X_{1}, \ldots, X_{N} = \text{lin. indep.}$

1. Each $X_{\alpha}$ is tangent to $M$.

$\nabla A \cdot \nabla A = A \nabla A - \nabla A \cdot J \nabla A_{\alpha} = \{A_{\alpha}, A_{\beta}\} = 0.$

Mod: Can only be true if each of the $A_{\beta}$ is constant in the dire of $X_{\alpha}$.

$0 = \{A_{\alpha}, A_{\beta}\} = \tilde{X}_{\alpha} \cdot \nabla A_{\beta} = - \nabla A_{\alpha} \cdot J \nabla A_{\beta} = - \{A_{\alpha}, A_{\beta}\} = 0.$

2. The $X_{\alpha}$ span the tangent plane to $M$.

3. $\text{CM}_{A} \cdot \nabla A = \tilde{X}_{\alpha} \cdot \nabla A = A \nabla A - \nabla A \cdot J \nabla A_{\alpha} = \{A_{\alpha}, A_{\beta}\} = 0.$

4. $\text{CM}_{A} \cdot \nabla A = A \nabla A - \nabla A \cdot J \nabla A_{\alpha} = \{A_{\alpha}, A_{\beta}\} = 0.$

$\tilde{X}_{\alpha} \cdot J \cdot X_{\alpha} = \nabla A_{\alpha} \cdot J \cdot J \cdot \nabla A_{\alpha} = \{A_{\alpha}, A_{\beta}\} = 0.$
\( t_k = \text{param. along orbits of } A_k. \)

E.g., \((H, l_2), \text{rot. sym.} \) 
\[ t_2 \circ H^3 = 0 \text{ says } l_2 \text{ count along } H \text{- orbits, } \]
\[ H \text{- count along } l_2 \text{ orbits. } \]

\( l_2 \)-orbits are just rotations about \( z \)-axis.

\[ \bar{r} = R_z(\theta) \bar{r}_0 \]
\[ \bar{p} = R_z(\theta) \bar{p}_0 \]

Here \( t = \theta. \)

So, we see that the fact that these counts are in involution, means that
they are all count. along each others orbits. Draw orbits.

Ok, an important fact about all these flows along all these
different vector fields is that they all commute.

**Flows Commute.**

Here's what I mean by that. Pick, i.e., call it \( z_0. \)

\[ \bar{z}_2 \xrightarrow{t_1} \bar{z}_1 \xrightarrow{t_2} \bar{z}_2 \]

Let flow under \( A_1, \) flow for parameter value \( t_1, \) get \( \bar{z}_1. \)
Write this as

\[ \bar{z}_1 = T_1(t_1) \bar{z}_0 \]

Or let \( z_0 \) flow for parameter \( t_2. \)

\[ \bar{z}_2 = T_2(t_2) \bar{z}_0. \]

Do in reverse order,

\[ \bar{z}_2 = T_2(t_2) \bar{z}_1, \quad \bar{z}_1 = T_1(t_1) \bar{z}_0. \]
Question is, is $Z_3 = Z_4$? Ans. is, if $A_1, A_2$ any old observables, then no, but if they Poisson commute, then yes. That is,

$$T_1(t_1) T_2(t_2) Z_0 = T_2(t_2) T_1(t_1) Z_0 \quad \forall t_1, t_2, Z_0$$

$$\text{if } [A_1, A_2] = 0,$$

Comment on usual stat, classical observables always commute.
Comment, PB may be const.

I think I will not prove this fact - I know some of you already knew it, and others may get it on homework.

Anyway, we see that our surface $M$ is not only an inv. LI, but it is also crisscrossed with $N$ commuting flows or vector fields.

Ok, now it becomes possible to invoke some topology, which ends up in the following then:

**Then:** If $M$ is bounded (i.e., compact), then it is an

**N-torus**

The detailed proof of this then is given by Arnold; I'm just going to outline the proof and argue for its plausibility.

Look at a 2-torus.
If you put hair on this, you can comb it 2 different ways.

In contrast to a sphere, which cannot be combed without leaving a twist where the vector field vanishes.