Summary

\[ H(x, y, z) = E \]  \hspace{1cm} (HJ)

\[ \nabla \cdot (\mathbf{p} \psi) = 0 \]  \hspace{1cm} (AT)

**Thm:** A Lagr. manifold is a subset of energy surface

\[ H(x, y, z) = E \] iff it is invariant.

Our first task today is to understand the geometrical meaning of the multi-dimensional HJ eqn.

Last time it was pointed out that a soln of the HJ eqn represents geometrically a LM which is a subset of the energy surface. This is just a simple matter of interpretation, based on looking at the eqn.

However, our theorem also tells us that every soln of the HJ eqn is an invariant Lagr. manifold, and conversely.

Therefore, what this eqn is crying out to say is, Find me an invariant LM. This, in a sense, is what it means, classically, that while particles of that E is at rest.

Before proving this theorem, I'd like to point out that it is non-trivial. In particular, I'd like to show that there are lots of invariant, N-dim. surfaces which are not a subset of the energy surface.

\[ H = E \] (3-dim.)

Work in 2 DOF.
So, here's a 2D surfacements but not a part of our energy surface. Therefore by the thin, it is not Lagrangian.

Ok, now to prove our thin. The proof will use some linear algebra.

As preliminaries for this, I need to tell you some things about the geometry of phase space.

The first important thing is that phase space has no concept of distance associated with it. This is unlike config, space, which may have such a concept.

No sense to talk about dit, btw. 2 (q,p) points. You could define such a measure of distance in any arbitrary manner, but you would find that your defn had no dynamical significance. It would also not be invariant under CT's.

In a space which does have a measure of distance, you can form dot products between vectors:

\[ x \cdot y = \sum_{\mu=0}^{n} x^\mu g_{\mu \nu} x^\nu = \tilde{x} \cdot g \cdot y. \]

And you say, \[ \tilde{x} \cdot g \cdot y = 0 \Rightarrow x \cdot y \text{ are orthogonal.} \]

But phase space has no metric, so you can't say when vectors are orthogonal. However, phase space does have the tensor T, which in many respects is like a metric tensor, except it's antisymmetric.
So, because of this analogy, it's useful to make the following defn:

If \( \mathbf{x}, \mathbf{y} = 0 \), then \( \mathbf{x}, \mathbf{y} \) are skew-orthogonal.

(\( \mathbf{x}, \mathbf{y} \) vectors in phase space)

Now you'll recognize that the defn of an LM is equiv. to saying that all vectors tangent to an LM are skew-orthogonal.

We also proved, this defn is invariant under CT's, axis skew-orthogonal.

Now, there's an important fact about skew-orthogonality, which I will convert into a lemma:

**Lemma.** Let \( \mathbf{X}(1), \ldots, \mathbf{X}(k) \) be \( k \) linearly independent vectors in phase-space which are skew-orthogonal to one another. Then \( k \leq N \).

(i.e. \( \mathbf{X}(i) \cdot J \cdot \mathbf{X}(j) = 0 \))

That is, the maximum # of lin. indep. skew orth. vectors you can have is \( N \), the dim. of the space.

Here's the proof of the lemma:

Let \( \mathcal{V} = \text{span} \left\{ \mathbf{X}(1), \ldots, \mathbf{X}(k) \right\} = k \)-dim.

Let \( \mathcal{JV} = \text{span} \left\{ J\mathbf{X}(1), \ldots, J\mathbf{X}(k) \right\} = \text{also} k \)-dim.

because \( J \) is nonsingular matrix.

Further, every vector in \( \mathcal{V} \) is \( \perp \) to every vector in \( \mathcal{JV} \), by hypothesis. So 2 subspaces, \( \perp \) to each other, have no point in common except the zero-vector. Therefore:

\( 2k \leq 2N \), \( k \leq N \).
Now go back to our original theorem and use this lemma.

Since this is iff, state it both ways.

**Theorem:** If a Lagrangian manifold is a subset of energy surface, then it is invariant.

Illustrate this with a picture.

![Diagram of a 3D energy shell with Lagrangian manifold](image)

Make sure to prove it is invariant. Look at flow vector. Tangent to $H=E$, because $E$ conserved. So pick $E$, then can think of flow vector inside energy surface. Q: Tangent to LM?

Suppose not.

Choose 2 vectors which are tangent to LM, say $X_1, X_2$.

We know, $\tilde{X}_1 \cdot \tilde{J} \cdot X_2 = 0$. 
also look at skew-scalar product below. $\hat{z}, \hat{X}(1), \hat{X}(2)$.

$$\hat{X}(1) \cdot \hat{J} \cdot \hat{z} = \hat{X}(1) \cdot \hat{J}^2 \cdot \frac{\partial H}{\partial \hat{z}} = - \hat{X}(1) \cdot \frac{\partial H}{\partial \hat{z}} = 0$$

by assumption that $LM \subset ES$. Therefore all 3 vectors $\hat{X}(1), \hat{X}(2), \hat{z}$ are skew-orthogonal.

But the lemma says they cannot be lin. indep.; therefore $\hat{z}$ must be an l.c. of $\hat{X}(1), \hat{X}(2)$, i.e., tangent to the LM. 

$\Rightarrow$ LM is invariant. QED.

Now do this the other way.

**Thm. (4)** If a lagr. manifold is invariant, then it is the subset of some energy shell $H=E$.

Here's the proof. Draw LM.

That means, pick any point of $L$, and flow vector is tangent to $L$ there.

Now pick any 2 l.i. vectors $\hat{X}_1, \hat{X}_2$ at same point. Then by hyp., $\hat{z}$ must be an l.c. of the $\hat{X}$'s:

$$\hat{z} = \sum_{i=1}^{N} c_i \hat{X}_i.$$
Further, since $L$ is an LM, we know
\[ \overline{\mathbf{X}}_i \cdot J \cdot \mathbf{X}_j = 0. \]

Now we can show that energy is conserved along both the directions $\mathbf{X}_1$, $\mathbf{X}_2$. Here's how.

\[ \overline{\mathbf{X}}_1 \cdot \nabla H = - \overline{\mathbf{X}}_1 \cdot J^2 \cdot \nabla H = - \overline{\mathbf{X}}_1 \cdot J \cdot \overline{\mathbf{X}}_1 \cdot \sum_{i} c_i \overline{\mathbf{X}}_i = 0. \]

So $H$ is constant along all directions tangent to $L$, and therefore $L$ is a subset of the energy shell.

So, our original claim is proved, and now we can see that the t-indep. HJ eqn. has solutions which represent invariant LMs and conversely. So how to find invariant LMs?

Well, there are lots of ways, which depend somewhat on whether you're looking a unbounded problems (like scattering) or bound problems (which have quantization).

Probably for next couple of days we'll be talking about how to find such solutions to the HJ eqn.

Begin with a construction, which is quite general, based on the SOS method.

How many are families with SOS method?
How to find invariant Leja manifolds

I. Separable Systems:
   A. Separate the HT Eqn.

II. Unbound Systems:
   a. Follow families of orbits from initial value surface.

III. Bound Systems:
   A. Integrable:
      Find counts of motion
   B. Near-integrable, regular region:
      Follow single orbit
      Perturbation theory
   C. Near-integrable, irregular region or non-integrable:
      1. No method known.
      2. Smooth invariant IM's don't exist.

IV. Other methods.

To summarize:
1) Separability is a very specialized method
2) Detailed work for majority of cases
3) Not a fundamental property

All of these methods can be assimilated under a single conceptual framework, which involves the SOS method, so I'll begin with that.
The SOS method is often used in theory and practice for 1\textsuperscript{st} dep. Ham. systems.

\[ H = H(\vec{E}, \vec{p}). \]

Invented by Poincaré, and basic idea is to use conservation of energy to reduce the dimensionality of the space you have to look at.

Of course original space has 2N dimensions, and if \( H \) is indep. of \( t \), then energy is conserved, and you can get rid of one dim. just by restricting consideration to the energy surface \( H = E \).

\[ H = E \quad (2N-1)-\text{dim.} \]

So if you only want to look at orbits with a single \( E \), say for \( N=2 \), you can make do with a 3D space, which is an improvement over the original 4D phase space.

One should point out, however, energy surface in general will be curved, often nontrivial topologically. So don't expect to use nice rectangular coordinate on it. Example: 2D SHO

\[ H = \frac{p_x^2}{2m} + \frac{1}{2} m \omega_x^2 x^2 + \frac{p_y^2}{2m} + \frac{1}{2} m \omega_y^2 y^2. \]

Apart from constants, looks like

\[ H(\vec{z}) \sim \sum_i z_i^2 \sim S^3 \]

So looks like 3D surface of a sphere or ellipsoid embedded in 4D space.
So, going to energy shell gets rid of 1 var. Since $H=E$, is only $(2N-1)$-dim, we only need $(2N-1)$ coords to specify points on it. What coors do we use? Write

$$H(x_1, \ldots, x_N, p_1, \ldots, p_N) = E.$$  

coords on energy shell extra not indep.

Arbitrarily choose first $(2N-1)$ of these. As for $p_N$, we can solve the above and express $p_N$ as a fun of the others.

$$p_N = p_N(x_1, \ldots, x_N, p_1, \ldots, p_{N-1}; E).$$  

Generally multiple branches.

Let's sketch this for $N = 2$.

Energy shell $H = E$ (3D)

Now define SOS as any slice thru this 3D space which is at const. $x_N$. Then we see:

$$\text{SOS} = (H=E) \cap (x_N = \text{const}) = (2N-2)\text{dim. surface}$$

Now it turns out that for many purposes you can work with the SOS instead of the full phase space, and thereby reduce the dimensionality by 2.