Summary: \( H(\vec{x}, i\kappa \nabla) \psi(\vec{x}) = E \psi(\vec{x}) \)

\[ \psi(\vec{x}) = A(\vec{x}) e^{iS(\vec{x})/\hbar} \]

\( H_{\text{eqn}}(\vec{x}, \nabla S) = E \quad (\text{HJ}) \)

\( \nabla \cdot (\rho \vec{v}) = 0 \quad (\text{AT}) \)

\[ \rho = \lambda^2, \quad \vec{v} = \frac{\partial H_{\text{eqn}}(\vec{x}, \nabla S)}{\partial \rho} \]

These are the equations for the wave representation of energy eigenfunctions, and they give rise to some very interesting questions in systems with >1 DOF.

However, today we'll begin with just a single DOF, because that's a good place to begin and because the solution there is quite trivial.

1 DOF: \( H_{\text{eqn}}(x, \frac{dS}{dx}) = E \)

Obvious thing to do is solve for \( \frac{dS}{dx} \).

\[ \frac{dS}{dx} = p(x, E) \quad \text{where } p(x, E) \text{ is inverse of } H(x, p) = E. \]

Then, solution of HJ eqn is trivial,

\[ S(x) = \int_{x'}^{x} p(x', E) dx' \]
Now look at this geometrically in phase space.

\[ H(x, p) = \text{const.} \]

This is what it looks like for an oscillator, like SHO; notice that this curve is actually 3 things:

1) Constant energy surface
2) An orbit
3) A Lagr. Manif.

Now, interpret for \( p = p(x, E) \). So we see that in general, \( p(x, E) \) is a multi-branched fn., and that therefore so is \( S \).

\[
S_b(x) = \int_x^\infty p_b(x', E) dx'.
\]

In QM problems, where you have \( T+V \) hamiltonians,

\[
\frac{\dot{x}^2}{2m} + V(x) = E,
\]

\[
p(x, E) = \pm \sqrt{2m(E-V(x))}
\]

So, 2 branches, opposite signs. This implies that curve is symmetric about \( x \)-axis.
More generally, curve can be more complicated.

Notice that the curve \( H=E \) doesn't have to be a closed loop.

This is an unbounded, or scattering problem. (Continuous spectrum)

Notice also that \( H=E \) doesn't have to be in one piece.
Ok, so soln of HJ eqn is easy, except for multiple branches. What about the ST eqn? It's also easy.

\[ \frac{d}{dx} (p \cdot v) = 0 \]

\[ p \cdot v = \text{const.} \]

\[ \rho(x) = \frac{1}{|v(x)|} \]

\[ A(x) = \frac{1}{|v(x)|^{1/2}} \]

\[ \sum \int \frac{a_b}{|v_b(x)|^{1/2}} e^{-\frac{i}{\hbar} \int p_b(x', \xi) dx'} \]

The coefficients \( c_b \) give an l.c. of terms from different branches; worry later about what they are.

Notice that \( \Psi \) diverges when \( v = 0 \); these are the turning points of the classical motion, also the caustics.

Here, caustics = turning points
There's another way to understand this amplitude formula. Pick any point on orbit, call it $z_0$.

\[ \begin{align*}
\text{Then label points on the orbit by the elapsed time.} \\
(\text{let } t = \text{coordinate along orbit.})
\end{align*} \]

Now, the $\Delta t$ equ represents conservation of particles, so if you take some interval $\Delta t$ around $t=0$ and at a later time, then # of particles is the same in both intervals.

This is saying that $P = \text{const. wrt. } t$,

or $dt = \text{invariant measure}$.

So therefore in the $x$ coordinates, we have

\[ p(x) dx = \text{const. } x dt, \]

\[ p(x) = \frac{1}{|v(x)|}, \text{ as before.} \]
3 ways:
1) Budy layer analysis
2) Analytic contin.
3) Major's method

Ok, now let's consider what is involved in determining the coefficients $c_\phi$, which will give the relative amplitudes and phases between branches. (Actually, we'll find $c_\phi$ = phase factor only.)

Basic idea is to follow theMajor's method, set

$$\Psi_E(p) = \sum \frac{c_{\phi}}{|p(p)|^{1/2}} e^{-\frac{i}{\hbar} \int x(p',E)dp'}$$

then use the FT and SPA to demand consistency between the two forms. It's clear that you won't be able to determine all the $c$'s or $\alpha$'s, because an eigenfun is determined only up to an overall normalization, but once we specify one of the $c$'s or $\alpha$'s, all the others will be determined.

Before we begin, let's establish some conventions for our $x$-space and $p$-space actions. Take one connected piece of our orbit, and pick a point on it.

![Diagram](attachment:image.png)
Now introduce a coordinate on the orbit; call it \( x \). This can be any coordinate that uniquely labels points, e.g., it might be the elapsed time.

Then define

\[
S(\lambda) = \int_0^\lambda p(\tau) \frac{dx(\tau)}{d\lambda} \, d\lambda,
\]

\[
T(\lambda) = -\int_0^\lambda x(\tau) \frac{dp(\tau)}{d\lambda} \, d\lambda.
\]

These functions are single-valued functions of \( \lambda \), as long as the orbit doesn't close on itself. If the orbit does close on itself, we can introduce a branch cut at \( \lambda = 0 \), so that they are still single-valued.

In this way, we can deal with functions w/o branch cuts. Also, a simple relation between them,

\[
S(\lambda) = x(\lambda) p(\lambda) - x(0)p(0) + T(\lambda)
\]

(Get by integration by parts.)

Now we use these single-valued \( S(\lambda) \), \( T(\lambda) \) to define the multi-branched actions, \( S_\lambda(x) \) and \( T_\lambda(p) \).

Define:

\[
S_\lambda(x) = S(\lambda) \quad \text{when} \quad x(\lambda) = x \quad p(\lambda) = p_\lambda(x)
\]
and \( T_b(p) = T(A) \) when \( \phi(A) = \phi \), \( x(A) = x_b(p) \).

Then we have,
\[
\begin{align*}
\frac{dS_b(x)}{dx} &= P_b(x) \\
\frac{dT_b(p)}{dp} &= -x_b(p)
\end{align*}
\]

We also have the following relation:

If \( x \)-branch \( b \) and \( p \)-branch \( b' \) overlap, then
\[
S_{b'}(x) = x_p - x_0 p_0 + T_{b'}(p)
\]
when \((x,p)\) are in the overlap region.

With these conventions, let's write,
\[
\begin{align*}
\Psi_E(x) &= \sum_b \frac{C_b}{|\hat{\gamma}_b(x)|^{1/2}} \ e^{\frac{i}{\hbar} S_b(x)} \\
\hat{x}_b &= \frac{\partial H}{\partial p}(x, P_b(x)) \\
\hat{P}_b &= -\frac{\partial H}{\partial x}(x_b(p), p)
\end{align*}
\]

Now let's determine the \( c's \) and \( d's \) by the Maslov method.

To do this, it helps to have a specific example. So... consider
Here we have 2 \( x \)-branches, 1,2, and only one \( p \)-branch.

Take \( \Psi_1(x) = \frac{1}{|\dot{x}_1(x)|^{1/2}} e^{\frac{i}{\hbar} \frac{1}{2} S_1(x)} \), i.e., \( a_1 = 1 \).

\[
\Psi_2(x) = \frac{a_2}{|\dot{x}_2(x)|^{1/2}} e^{\frac{i}{\hbar} \frac{1}{2} S_2(x)}
\]

\[
\Psi_1(p) = \int \frac{dx}{\sqrt{2\pi\hbar}} e^{-i p x / \hbar} \left[ \Psi_1(x) + \Psi_2(x) \right]
\]

Initially, choose \( p > p_c \), so \( (x,p) \) lies in 1. Do SPA just like we've always done, get

\[
\Psi_1(p) = \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{|\dot{x}_1(x)|^{1/2}} \sqrt{\frac{2\pi\hbar}{|dp/dx|}} e^{\frac{i}{\hbar} \left[ S_1(x) - p x \right]} e^{i \frac{\hbar}{2} \frac{1}{|\dot{x}_1(x)|} \frac{dp}{dx}}
\]

Notice for this value of \( p \), the \( \Psi_2 \) term doesn't contribute.
Now clean this up, we get (note, $\frac{dx}{dp} < 0$)

$$\psi_1(p) = \frac{e^{-i\pi/4}}{|\phi_1(p)|^{1/2}} e^{\frac{i}{\hbar} [T_1(p) - xo_0]}$$

Now this was derived for $p > p_0$, but we can argue by continuity that this must also be valid for the rest of the $\# 1$ $p$-branch.

So now we know the $\# 1$ $p$-branch where it overlaps with the $\# 2$ $x$-branch, so we can FT back to get the $\# 2$ $x$-branch.

$$\psi(x) = \psi_1(x) + \psi_2(x) = \int \frac{dp}{\sqrt{2\pi\hbar}} e^{+ipx/\hbar} \psi_1(p)$$

$$= \frac{1}{\sqrt{\phi_1(x)}} e^{\frac{i}{\hbar} S_1(x)}$$

$$+ \frac{1}{\sqrt{2\pi\hbar}} \left( e^{-i\pi/4} \sqrt{\frac{2\pi\hbar}{|\phi_1(p)|}} \right) e^{\frac{i}{\hbar} [T_1(p) - xo_0 + x] \sqrt{-\frac{dx}{dp}}}$$

$$\times e^{-i\pi/4 \text{ sgn} \frac{dx}{dp}}$$

where $(x,p)$ in and turn lie on $x$-branch $\# 2$.

Clean this up, we get, note, $\frac{dx}{dp} > 0$.
\[ \psi_c(x) = \frac{1}{|\dot{x}_1(x)|^{1/2}} e^{i \frac{i}{\hbar} S_1(x)} + \frac{e^{-i \pi/2}}{|\dot{x}_2(x)|^{1/2}} e^{i \frac{i}{\hbar} S_2(x)} \]

So, you see that the lower \( x \)-branch has a Maslov index of 1 relative to the upper branch.

If you continue in this way, switching back and forth between branches, you can find the Maslov indices for an orbit of any complexity, and, naturally, the result will be,

\[ \psi_c(x) = \sum_n e^{i \frac{i}{\hbar} S_n(x) + \pi \hbar/2} \]

You can also make some simple rules for the Maslov phase shifts. Reason we get \( \Delta \mu = 1 \) for our problem has was \( \frac{dp}{dx} < 0 \) on upper branch, \( \frac{dx}{dp} > 0 \) lower.

Tangent line turns clockwise.

Gives a rule:

**Rule for Maslov index in 1D:**

Tangent line clockwise through \( x \)-caustic, \( \Delta \mu = +1 \);

"counterclockwise", \( \Delta \mu = -1 \).

Applies to LM's in 1DOF problems.