\[ G(x, t; \xi_0, t_0) = \int \frac{d^N p}{(2\pi \hbar)^N} e^{2\sqrt{\hbar} \int_{t_0}^t \frac{\hbar}{2m} \left[ T_m(p(t'; \xi_0, t_0) - p_0 x_0 + \frac{p_0^2}{m} \right] \right] \]

Since I said some things in Friday's lecture that were wrong, we'd better begin today by straightening them out. (Thank you, Chi)

\[ \xi(t) = \xi_0 + \frac{p(t)}{m} t + O(t^2). \]

Last time we had determined that finding the Green's function comes down to doing this integral by stationary phase, and that in turn comes down to answering the question, 'how many orbits?', since those orbits precisely give us the saddle points of the integral.

What was stated incorrectly on Friday was the claim that for short enough times, there is always a single orbit. In fact, it's possible to give examples where there is only a single orbit, but also others where there is more than one.

Then of Jacobi: reason I couldn't find reference...

So, look at examples. First, let's look at Hamiltonians which are at most quadratic polynomials in \( q, p \).

\[ H(q, p) = \frac{1}{2} (q, p) \left( \begin{array}{cc} K_{qq} & K_{qp} \\ K_{pq} & K_{pp} \end{array} \right) \left( \begin{array}{c} q \\ p \end{array} \right) + (B_q, B_p) \left( \begin{array}{c} q \\ p \end{array} \right) \]
This includes a number of systems of physical interest. Examples:

\[ H = \frac{p^2}{2m} \quad \text{free particle} \]

\[ H = \frac{p^2}{2m} + mgx \quad \text{uniform grav. field} \]

\[ H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \quad \text{Harmonic oscillator} \]

\[ H = \frac{1}{2m} \left( \vec{p} - \frac{e}{c} \vec{A}(x) \right)^2 \quad \vec{A} = \frac{1}{2} \vec{B} \times \vec{x} \]

\[ H = xp_y - yp_x \quad \text{Uniform B-field, rotations about } \z \text{-axis.} \]

The significance of all these cases is that the classical, laws of motion are \textcolor{red}{linear}. Rewrite:

\[ H(z) = \frac{1}{2} \vec{z} \cdot \vec{K} \cdot \vec{z} + \vec{B} \cdot \vec{z} \]

\[ \dot{z} = \nabla \frac{\partial H}{\partial \dot{z}} = \vec{J} K \cdot \dot{z} + \vec{J} \cdot \vec{B} \]

\[ \Rightarrow \quad z(t) = \vec{S}(t) \cdot \vec{z}_0 + \vec{A}(t). \]

So, the relation between initial, final, and linear, and linear mappings, always take straight lines into straight lines.

Therefore, \( z \), no matter what \( t \) we look at, is always a line, and it either gives a unique orbit or else is vertical.

Now look at another example, an oscillator like the Morse oscillator, which is often used to model forces between atoms in molecules.
Almost to turn on time. Last time we went through an analysis of what happens to $L_E$ for small $t$, showed that still looks like straight line. This analysis was correct, but only for small enough values of $p$. Morse oscillator illustrates this.

$p > 0$, large, $L_E$ looks like straight line.

$p < 0$, large, particle goes to the left, hits hard wall, bounces off. $p$ changes sign.

Finally, take another example, quartic oscillator.

$V(x) = x^4$.

Rises rather rapidly; for large $p$, looks like a hard wall.
So in this case,

So here we have an \( \infty \) # of branches after arbitrarily short times.

OK, so let's try and make a statement about \( G \), that's correct.

\[
G(x,t; x_0, t_0) = \sum_{\text{orbits}} \frac{e^{-i\mathcal{H}t/\hbar}}{(2\pi \hbar)^{3/2}} \left| \det \frac{\partial^2 S_{Hf}}{\partial \vec{x} \partial \vec{x}_0} \right|^{1/2} e^{\frac{i}{\hbar} S_{Hf}(\vec{x}, t; \vec{x}_0, t_0)} e^{-i\mu \pi t/2}
\]

for \( H = p^2/2m + V(x, t) \)

where \( \mu = 0 \) on small \( t \), small \( \phi \) branch.

Now examples. Mention free particle. Exact result.
Another important example is the SHO.

\[ H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}. \]

Homework, \( S_H = \frac{m\omega}{2} \frac{(x^2 + \dot{x}^2) \cos \omega t - 2\dot{x}x_0}{\sin \omega t} \)

Initially examine this for \( t > 0, \) small, not too large.

\[ \frac{\partial S_H}{\partial x \cdot \partial x_0} = \frac{m\omega}{\sin \omega t} \]

\[ G(x,t;x_0) = e^{-i\pi/4} \sqrt{\frac{m\omega}{\sin \omega t}} e^{i\frac{m\omega}{2\pi} \frac{(x^2 + \dot{x}^2) \cos \omega t - 2\dot{x}x_0}{\sin \omega t}} \]

\( 0 < t < \pi/\omega. \)

Now, examine range of validity?

Valid up to \( \frac{1}{2} \) cycle.

Also exact result.
ok, now we're done with t-dep. WKB theory, and we turn to the problem of t-indep. theory, which usually means energy quantization in QM.

\[ H \psi = E \psi. \]

For other wave eqns, it means:

\[ \psi = \text{const}. \]

For example, the problem of the lens in optics is like this:

\[ \exp(-ikx - \omega t) \]

Before we get into this, I'd like to point out that the t-indep. problem is actually more general, and includes the t-dep. problem as a special case. The easiest way to see this is to go to CM, and summon up a common trick for dealing with t-dep. systems.

Say we have \( H(x,p,t), \ N \) DOF.

Can map this into an autonomous Ham. sys. of \((N+1)\) DOF, by making energy, time conjugate variables. Just write down a super-Hamiltonian,

\[ \tilde{H}(x,p,t,h) = H(x,p,t) + h \]

where \( h \) is conjugate to \( t \). Let \( t \) = parameter of orbits, then
\[ \frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial p}, \quad \frac{dt}{dr} = \frac{\partial H}{\partial x} = \frac{\partial H}{\partial x} + 1 \]

So this gives \( t = \tau \). Also, since \( \partial H \) is independent of \( \tau \), it is conserved along orbits:

\[ \frac{dH}{dt} = \frac{dH}{dr} + \frac{dt}{dr} = \frac{dH}{dt} - \frac{\partial H}{\partial \tau} = 0. \]

Easy to extend this idea to QM: just interpret \( \hbar \) as the operator,

\[ \hbar \rightarrow -i\hbar \frac{\partial}{\partial t} \]

just like \( p \rightarrow -i\hbar \frac{\partial}{\partial x} \).

Then \( H \psi = E \psi \) is quantization for some Hamiltonian

\[ H(x, -i\hbar \frac{\partial}{\partial x}, t, -i\hbar \frac{\partial}{\partial t}) \psi \]

\[ = H(x, -i\hbar \frac{\partial}{\partial x}, t) \psi - i\hbar \frac{\partial}{\partial t} \psi = E \psi. \]

So, if we can solve this for \( E = 0 \), it gives us the soln of the original time-dependent problem.

Mathematically, this is all trivial, and the only reason for separating the 2 cases in practical considerations (and pedagogical).

But note: this switching of roles of space, time vars is exactly what you do for an optics problem; make \( Z \) look like \( t \).
So anyway, look at our quantization problem.

\[ H(\vec{x}, -i\hbar\nabla) \psi(\vec{x}) = E \psi(\vec{x}) \]

Now assume \( H \) is independent of \( t \).

Ok, we can analyze this by the WB ansatz, except now we don't have any \( t \)-dependence; set

\[ \psi(\vec{x}) = A(\vec{x}) e^{\frac{i}{\hbar}S(\vec{x})} \]

plug in, expand, leave details as exercise; get

\[ H(\vec{x}, \nabla S) = E \quad (\text{HJ}) \]

\[ \nabla \cdot (\vec{p} \hat{v}) = 0 \quad (\text{AT}) \]

where \( \vec{p} = \vec{A}^2 \)

\[ \vec{v} = \frac{2H}{\vec{p}} (\vec{x}, \nabla S) \quad \text{(general)} \]

\[ = \frac{1}{m} \nabla S \quad (H = \frac{\vec{p}^2}{2m} + V) \text{,} \]

Notice that defn of \( \vec{v} \) is basically obtained from classical Ham. eqns.

Note also (could have mentioned earlier) \( \vec{v} \) which occurs here is group velocity of waves, not phase velocity;

\[ H(\vec{x}, \vec{p}) = E = \hbar \omega(\vec{x}, \vec{p}) \]

\[ \vec{p} = \hbar \vec{\omega} \quad \vec{v} = \frac{\vec{\omega}}{\partial \vec{p}} \text{.} \]
Something else to mention. Plot wave fronts in config space.

\[ \vec{p} = \nabla S = \perp \text{ to wave fronts.} \]

\[ S = \text{const.} \rightarrow x \]

But \( \vec{p} \) of particles not in general. \( (T+V \text{ Ham is an exception.}) \)

\[ e.g., \text{ magnetic field, } \vec{U} = \frac{1}{m} \left[ \nabla S - \frac{\vec{E}}{c} \right]. \]

Anyway, before we do the general case, let's look at 1D problems, where it's especially easy to solve the HJ eqn.

1D: \[ H(x, \frac{dS}{dx}) = E. \]

Obviously, solve for \( \frac{dS}{dx} \),

\[ \frac{dS}{dx} = \phi(x, E). \]

\[ S(x, E) = \int_x^x \phi(x', E) \, dx'. \]

Look at this geometrically.

\[ H(x, \vec{p}) = E. \]