Summary:

\[
\int \sum_s \int \frac{f(z)}{e^z} \, dz \, e^{\frac{z-2z_s}{\sqrt{\varepsilon}}} = \sum_s \int \frac{f(z)}{e^z} \left( \frac{1}{2} f''(z_s) (z-z_s)^2 + \frac{1}{24} f''''(z_s) (z-z_s)^4 + \cdots \right) \, dz
\]

\[
\frac{z-2z_s}{\sqrt{\varepsilon}} = y.
\]

Last time we were just putting the finishing touches on the SPA, by considering the issue of higher order approximations.

We argued that as \( \varepsilon \to 0 \), the region in the \( z \)-plane around the saddle points which contributes most to the integral must shrink as \( \sqrt{\varepsilon} \). Therefore the substitution is logical, and it gives

\[
\sum_s \frac{f(z_s)}{\sqrt{\varepsilon}} \int \frac{1}{e^z} \left( \frac{1}{2} f''(z_s) y^2 + \frac{\varepsilon}{6} f''(z_s) y^3 + \frac{\varepsilon}{24} f''(z_s) y^4 + \cdots \right) \, dy
\]

In general, can't do an integral that involves some cubic or quartic polynomial in \( y \). But, because of our scaling, we see that over the region of interest, these terms are small, so we expand them in power series in \( y \).

\[
\varepsilon^{\frac{1}{6}} f''(z_s) y^3 = 1 + \frac{\varepsilon}{6} f''(z_s) y^3 + \frac{\varepsilon}{72} f''''(z_s) y^6 + \cdots
\]

\[
\varepsilon^{\frac{1}{24}} f^{(4)}(z_s) y^4 = 1 + \frac{\varepsilon}{24} f^{(4)}(z_s) y^4 + \cdots
\]

Now, the most important feature of this calculation is that all odd powers of \( y \) will vanish on integration.
But note: odd powers of $y$ are the only ones involving half integral powers of $\epsilon$. So you get a power series in $\epsilon$, involving only integral powers. The coefficients can all be evaluated in terms of $\Gamma$ for:

$$\int_{-\infty}^{\infty} dy \ y^{2n} e^{-\alpha y^2}.$$ 

Anyway, the most important conclusion for us is,

$$\int_{\epsilon} d\epsilon \ e^{\frac{f(\epsilon)}{\epsilon}} = \sum_{5} \sqrt{\frac{2\pi \epsilon}{-f''(\epsilon)}} e^{f(\epsilon) / \epsilon} \left[ 1 + O(\epsilon) \right].$$

Notice what this does in WKB theory, where we identify $\epsilon$ with $h$. SPA gives answer valid to within $O(h)$ corrections.

Mention: Series not converged.
Finally, there's one more important case, namely when you have an integral,

\[ \int \frac{f(z)}{\epsilon} \, \xi \, A(z) \, e \]

Again, you find saddles of \( f(z) \), expand everything about it, including \( A \).

\[ L_{1} A(z) = A(z_{3} + \epsilon \xi y) \]

\[ = A(z_{3}) + \epsilon \xi A'(z_{3}) y + \frac{\epsilon^{2}}{2} A''(z_{3}) y^{2} + \ldots \]

This shows 1st term again cancels, and 2nd term causes corrections only at \( O(\epsilon) \). Therefore,

\[ \int \frac{f(z)}{\epsilon} \, \xi \, A(z) \, e = \sum s \sqrt{-f''(z_{3})} \, A(z_{3}) \, e \left( 1 + O(\epsilon) \right) \]

Ok, now let's go through an important example of the SPA.

This is the Airy fn. In order to motivate this with a physical example, let's consider the QM problem of a particle in a uniform gravitational field.

\[ H \psi = E \psi \quad \quad H = \frac{p^{2}}{2m} + mgx. \]

\[ -\frac{\hbar^{2}}{2m} \frac{d^{2} \psi}{dx^{2}} + (mgx - E) \psi = 0. \]
First we scale this to get rid of the physical constants.

Let \( x = x_0 + a z \). Do a linear transform.

\[
x_0 = \frac{E}{m g},
\]

\[
a = \left( \frac{\hbar^2}{2m^2 g} \right)^{1/3}
\]

Sketch what you expect the solution to look like.

\[
\frac{d^2 \psi}{dz^2} - z \psi = 0. \quad \text{This is std. form for Airy's eqn. and general solution is}
\]

\[\psi = a \text{Ai}(z) + b \text{Bi}(z).\]

It's fairly easy to get an integral representation for \( \text{Ai}(z) \).

Just consider FT of \( \psi \), like going to momentum space.

\[
\psi(k) = \int dx \, e^{-ikx} \psi(x).
\]

\[-k^2 \psi - i\frac{d\psi}{dk} = 0. \quad \frac{d\psi}{dk} = ik^2 \psi
\]

\[
\psi(k) = ce^{ik^{3/3}}
\]

\[
\psi(z) = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{i(kz + k^{3/3})} \right] = \text{Ai}(z)
\]
what happened to Bi(\(\tilde{z}\))? Only have a 1st order ODE in \(k\)-space, so only got one lin. indep. soln.

Answer is, Bi(\(\tilde{z}\)) is also given by this integral, but with a different contour of integr.

So let's evaluate this integral by SPA. Note the change of notation — var. of integr. is now \(k\), not \(z\); \(z\) is merely a parameter of integral. So, we'll be doing contours in \(k\)-plane.

To begin, let's take \(z\) to be positive.

\[ z = \text{real, pos.} \]

\[ \int dk \in C, \quad f(k) = i \left( k^2 + k^3/3 \right). \]

First find the saddles. \( f'(k) = i \left( z + k^2 \right) = 0, \)

\[ k_s = \pm i \sqrt{z} \]

2 saddles, both on Im axis.

\[ \lim k \to \pm \frac{\pi}{2} \]

\[ \text{Re}(k) \]

\[ \text{Im}(k) \]

\[ R = -\pi/6 \]

\[ R = 5\pi/6 \]
Now let's get an idea of the range and valley structure of the integrand. First look at large $k$.

\[ \text{large } k, \quad f(k) \sim \frac{i k^3}{3} = \frac{e^{3i\theta}}{3}, \quad \text{(large } k). \]

Let $k = r e^{i\theta}$. Observe $\lim_{k \to \infty} f(k) \to \infty$, but to distinguish range, valley, need sign.

\[ \text{Re } f(k) = \frac{1}{3} r^3 \cos(3\theta + \pi/2). \]

So, if $\cos > 0$, goes to $+\infty$; range.

$< 0$, goes to $-\infty$; valley.

\[ \cos(3\theta + \pi/2) = \begin{cases} +1 & \text{Range} \\ -1 & \text{Valley} \end{cases} \]

Range: $3\theta + \pi/2 = 0, 2\pi, 4\pi, \ldots$

$3\theta = -\pi/2, \frac{3\pi}{2}, \frac{7\pi}{2}$

$\theta = -\pi/6, \frac{\pi}{2}, \frac{7\pi}{6}, \ldots$

Valley, $3\theta + \pi/2 = \pi, 3\pi, 5\pi$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}, \ldots$
Now, these are only the asymptotic ranges and valleys, they are distorted near the origin by the term $kZ$.

So let's look at what happens in neighborhood of saddle. Look at upper saddle first.

\[ k_s = +i\sqrt{Z}. \]

Expand $f(z)$ about this,

\[ f(z) = i \left( kz + \frac{k^3}{3} \right) \]
\[ f'(z) = i \left( z^2 + k^2 \right) \]
\[ f''(z) = 2i \frac{z}{k} \]

\[ f(k_s) = i \left[ (i\sqrt{Z})(\sqrt{Z}) + \frac{1}{3} (i\sqrt{Z})^3 \right] = -\frac{2}{3} i \sqrt{Z}^{3/2}. \]

\[ f'(k_s) = 0 \]
\[ f''(k_s) = -2i\sqrt{Z}. \]

So, \[ f(z) \approx -\frac{2}{3} i \sqrt{Z}^{3/2} - \sqrt{Z} (k - k_s)^2. \]

Now find directions of steepest ascent, descent contours.

From last time,

\[ W_+ = \sqrt{\frac{|f''(z_s)|}{f'(z_s)}} \quad \sigma = \sqrt{-1} = \pm i \]
\[ W_- = \sqrt{\frac{|f''(z_s)|}{-f'(z_s)}} = \sqrt{1} = \pm 1. \]
Do this with other saddle, you get \( W_+ = \pm 1 \)
\( W_- = \pm i \)

Now you can just guess where the steepest descent contours go.

And you see, our original contour can be distorted to go over upper saddle, but not lower one, with

\[
W_- = +1
\]

Therefore

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i(kz + k^3/3)}}{-\infty} = \frac{1}{2\pi} \sqrt{\frac{2\pi}{2 \sqrt{z}}} e^{-\frac{2}{3} z^{3/2}}
\]

Or

\[
Ai(z) \approx \frac{1}{4\pi^{1/4}} e^{-\frac{2}{3} z^{3/2}} z \text{ real, } > 0
\]

A+S, Eq. 10.4.59.
Ok, now let's take another case, where $z$ is real and negative.

Now we find,

$$k_s = \pm \sqrt{-z} = \text{real}$$

Take $k_s = +\sqrt{-z}$ further.

$$f(k_s) = \mp \frac{2}{3} i (-z)^{3/2}$$

$$f'(k_s) = 0$$

$$f''(k_s) = \pm 2 i \sqrt{-z}$$

$$W_+ = \sqrt{\frac{|f''(k_s)|}{f''(k_s)}} = \frac{1}{\sqrt{i}} = \pm e^{-i\pi/4} \quad \text{(Right saddle)}$$

$$W_- \text{ must be orthogonal} = \pm e^{+i\pi/4}$$

Similarly for other case. Sketch in ranges, markers.

Now contour passes over both saddles.
So,

\[ \frac{1}{2\pi} \int \frac{dk}{k} e^{i(2\pi kz + k^{3/2})} = \frac{1}{2\pi} \left\{ e^{\frac{i\pi}{4}} \sqrt{\frac{2\pi}{2\sqrt{1 - z^2}}} e^{\frac{2}{3}i(-z)^{3/2}} \right. \]

[Diagram of a function labeled \( A_i(z) \).]

\[ \frac{1}{\sqrt{\pi}} \frac{1}{(-z)^{1/4}} \cos \left[ \frac{2}{3}(-z)^{3/2} - \frac{i\pi}{4} \right] = A_i(z) \quad \text{z real, <0.} \]

Case \( z \to 0 \),

Analytic Contin.