Today we begin the subject of the stationary phase approximation.

To motivate this, let's pose the problem of finding the WKB wave function in momentum space. There are 2 approaches to this that come to mind.

First, we might write down the Sch. eqn. in momentum space,

\[ H(x, p, t) = \text{QM Hamiltonian} \]

\[ H(\xi \frac{\partial}{\partial \xi}, p, t) \psi(p, t) = i \hbar \frac{\partial \psi}{\partial t}(p, t). \]

Then plug in WKB ansatz in momentum space,

\[ \psi(p, t) = B(p, t) e^{\frac{i}{\hbar} T(p, t)} \]

Expand this out, find some kind of HT and PT eqns, and solve them.

Later on will actually take this approach, and discuss it in more detail.

In the other approach, we start with the WKB wave fn. in \( \xi \)-space, which we already know, and do a FT.

\[ \psi(p, t) = \int \frac{dx}{\sqrt{2\pi \hbar}} e^{-i p x / \hbar} A(x) e^{i \frac{\hbar}{\pi} S(x, t)} \]

Now in general, we can't do this integral exactly. But since the WKB wave fn. isn't exact — it's only valid up to \( O(\hbar) \) — it makes sense to look for an approximation to this integral which is of the same order of approx. so in the wave fn.

And that leads us to the SPA — its main ingredient is that the integrand is rapidly oscillating.
So we'll approach this as a mathematical problem.

Suppose we have

\[ \int_C e^{f(z)} \, dz \]

where \( z = x + iy \) is a complex variable.

Of course, we can deform the contour.

We begin by looking at where in the complex plane the integrand is large or small. Write

\[ f(z) = u(x,y) + i\, v(x,y), \]

so

\[ |e^{f(z)}| = e^{u(x,y)}. \]

Therefore, the integrand is a maximum, where \( u(x,y) \) is a maximum.

Note, \( u, v \) are not arbitrary, but satisfy Cauchy conditions,

\[ u_x = v_y, \]

\[ u_y = -v_x. \]

These imply

\[ \begin{cases} u_{xx} + u_{yy} = \nabla^2 u = 0 \quad \text{and} \quad \nabla^2 v = 0, \end{cases} \]

also, \( u_x v_x + u_y v_y = \nabla u \cdot \nabla v = 0. \)

Now, a maximum of \( u \) can only occur where

\[ u_x = u_y = 0. \]
But this condition doesn't guarantee a maximum; it could be a min or a saddle. Which one? To find out, must look at matrix of 2nd derivs.

\[
M = \begin{pmatrix}
  u_{xx} & u_{xy} \\
  u_{yx} & u_{yy}
\end{pmatrix}.
\]

Let \( \lambda_1, \lambda_2 \) = eigenvalues of \( M \). Note, they must be real, since \( M \) is symmetric.

Then:
- \( \lambda_1, \lambda_2 > 0 \): min.
- \( \lambda_1, \lambda_2 < 0 \): max.
- \( \lambda_1 > 0, \lambda_2 < 0 \): saddle
- \( \lambda_1 < 0, \lambda_2 > 0 \): saddle
- \( \lambda_1, \lambda_2 = 0 \): no info.

Ok, so the secular eqn. of \( M \) is

\[
x^2 - (\text{Tr}M)x + \text{det}M = 0.
\]

But, \( \text{Tr}M = u_{xx} + u_{yy} = \nabla^2 u = 0 \). Therefore,

\[
\lambda = \pm \sqrt{-\text{det}M} \quad \text{(must have \( \text{det}M < 0 \)).}
\]

(Let's exclude the case \( \text{det}M = 0 \), i.e., \( \lambda_1 = \lambda_2 = 0 \), as atypical.)

So this shows an important fact, that \( u(x,y) \) does not have any maxima or minima, only saddles.

So let's call root of \( u_x = u_y = 0 \) saddle points.

Now, turns out you can formulate the condition for a saddle point without resorting to splitting \( f \) into its real and imag. part. Cauchy \( \Rightarrow \)

\[
v_x = v_y = 0, \quad \text{i.e.,} \quad \frac{df}{dz}(z) = 0
\]

Saddle points are roots of \( f \).
(Mark them in complex plane).

1. Mountain ranges, valleys.
2. Traveller.
3. Steepest contours.
4. Orthogonal at saddles.
5. Eigenvectors. Note: \( z = k' \).
6. \( x = \text{const. on steepest descent} \).

Now the idea of the SPA is to deform the contour just like you would do if you were a traveller going from A to B, and you wanted to minimize the amount of mountain climbing you'd have to do.

At each saddle,

Now it's possible to formulate the basic idea of the SPA. Namely, we try to deform our contour so that it passes over these saddle points, and also so that the saddles represent the maximum values of \( u(x,y) \) on the contour.

If you can do that, then the value of the integral will be dominated by regions around the saddles.

Ok, this saddle point structure can be used to divide the complex plane up into mountain ridges and valleys. Idea is, pick a saddle point, and follow the direction in which \( u \) is increasing most rapidly. This is a steepest ascent contour, can see it on diagram.

Similarly, steepest descent is direction where \( u \) is decreasing most rapidly.
Can write down an equation for these.

\[ \frac{d^2x}{d\mu} = -V_n \quad \text{(ascent)} \quad \dot{x} = \left( \frac{x}{5} \right) \]

\[ \frac{d^2x}{d\mu} = -V_n \quad \text{(descent)} \quad (\mu = \text{a parameter}) \]

Must make these subject to i.e.?s \((x_0, y_0) = \text{saddle}\).

Now, you can almost believe from looking at the diagram, that the steepest asc. + des. contours cross at right angles at a saddle. Look at it from above:

\[ \begin{array}{c}
\uparrow \\
\rightarrow \\
\downarrow \\
\leftarrow
\end{array} \]

\( \lambda > 0 \) \quad \lambda < 0 \,

\text{arrow = downhill direction}

\text{These 2 directions are \(-1\) because eigenvectors of the 2nd deriv. matrix,}

\text{Follow these for longer distance, contours can bend around.}

In a practical problem, it may not be easy to integrate these eqns and find out what the SD contours are; may have to do it on a computer.

Now, an important fact about these contours comes from

\[ \nabla u \cdot \nabla v = 0. \]

This shows that \( v = \text{const. on steepest contours} \).

Therefore \( v(x) = v(x_0) \) \( \Rightarrow \) \( v(x) = v(x_0) \).

But \( v \) is the Im. part of \( f \), \( f(x) = v(x) + iu(x) \).
So this shows that the phase of the integrand is constant along the steepest descent contours. This is the reason for the term SPA.

Ok, let's suppose we have determined the saddle points and the steepest descent contours, and we find that our original contour can be deformed to pass over one of them.

Therefore we know the value of the integral is dominated by those parts of the contour in the neighborhood of the saddle points. How can we use this fact to approximate?

Well, we just Taylor expand our fn. \( f \) in neighborhood of \( z_0 \).

\[
\oint_{C} f(z) \, dz 
\approx \oint_{\text{saddle}} f(z_0) + \frac{1}{2} f''(z_0) (z-z_0)^2 
\]

Gives a Gaussian integral. Look at one of these integrals.

Draw downhill arrows again.

Let \( w = \) normalized eigenvector of \( \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \) at saddle with negative eigenvalue, pointing in direction of contour.

\[ |w| = 1. \]
Then we change variables:

\[ z = z_0 + W s, \quad s = \text{real distance.} \]

\[ dz = W ds \]

\[
\int w ds \quad e^{f(z_0) + \frac{1}{2} f''(z_0) W^2 s^2}
\]

...constant factors.

Now, it turns out, I'll leave it as an exercise, that

\[ f''(z_0) W^2 = -\lambda, \quad \lambda = \text{pos. eigenvalue}. \]

(say a little about how you get this).

Therefore

\[
\int ds \quad e^{-\frac{\lambda}{2} s^2}
\]

\[ \text{a purely real integral} \]

\[
= \sqrt{\frac{2\pi}{\lambda}} w e^{f(z_0)}
\]

Therefore,

\[
\int_{C} dz \quad e^{f(z)} \sim \sum_{\text{saddle}} \sqrt{\frac{2\pi}{\lambda_s}} w_s e^{f(z_s)}
\]
Ok, there's an easier way to get the answer, that most people use. In this you just say,

$$\int dz \, e^{f(z)} \approx \sum_{s} \int dz \, e^{f(z)} \, e^{-\frac{\Delta z}{2}} \Delta z$$

It's easy to show this is consistent; just note,

$$f''(z) \Delta z^2 = -\lambda$$

$$\Delta z = \pm \sqrt{-\frac{\lambda}{f''(z)}}$$

Note, |f''(z)| = \lambda.

However, there's a problem with the easier formula— you're taking the $n^{th}$ of a complex number, and you don't know which branch to take. This is obviously important, because if you reversed the direction of the contour, the integral would change sign.

Ok, sometimes we're interested in higher order approximations to SPA. For this purpose, it helps to introduce an ordering parameter $\epsilon$.

So consider

$$\int dz \, \frac{f(z)}{\epsilon}$$

In WKB theory, identify $\epsilon$ with $\hbar$. Now expand, get

$$\int dz \, e^{\frac{1}{\epsilon} \left[ f(z) + \frac{1}{2} f''(z) \Delta z^2 + \frac{1}{6} f^{(3)}(z) \Delta z^3 + \ldots \right]}$$

If we just take the first 2 terms, the analysis above shows that the main contributions come from a region of distance

$$|\Delta z| \sim \sqrt{\frac{\lambda}{\epsilon}}$$
around the paddle. So, set

$$\frac{x - 2s}{y} = y', \quad dz = y' \sqrt{\lambda} dy$$

and get

$$\sqrt{\frac{\lambda}{\pi}} \int dy \exp \left[ \frac{1}{2} f'' y'^2 + \frac{\lambda}{6} f''' y^3 + \frac{\lambda}{24} f''''(y) y^4 + \ldots \right]$$

This shows that all the higher order corrections are small, so you can expand them exponentials out in Taylor series.

$$\sqrt{\frac{\lambda}{\pi}} e^{f''(y) y^2}$$

$$= e^{-\frac{1}{24} + \frac{1}{72} = \frac{3}{24} + \frac{1}{18}}$$

$$= 1 + \frac{\lambda}{6} f''' y^3 + \frac{\lambda}{72} f'''' y^4 + \ldots$$

Now do the Gaussian integral, and you see the $y^3$ terms cancel (odd integrand). First correction comes from $y^4$; this is a moment of a Gaussian, and is double. This shows,

$$\int dz \ e^{f(z)/\lambda} = \sqrt{-\frac{2\pi \lambda}{f''(z_0)}} \ e^{f(z)/\lambda} \left( 1 + O(\epsilon) \right)$$