Summary:

Thus: A transformation

\[ Q = Q(q,p,t) \]
\[ P = P(q,p,t) \]

is canonical iff for every \( H(q,p,t) \) there exist functions \( K,F \) such that

\[ pdq - H dt = P dQ - K dt + dF \]

1. \((q,Q)\) are independent. Let \( F = F_1(q,Q,t) \)

\[ \phi = \frac{\partial F}{\partial q}, \quad P = -\frac{\partial F}{\partial Q} \]
\[ K = H + \frac{\partial F_1}{\partial t} \]

Now if \((q,Q)\) are not independent, then it will always be possible to find some pair of old and new variables that are.

So suppose \((q,P)\) are independent. This leads us to Case 2.

Case 2. \((q,P)\) independent.

Let \( F = F_2(q,P,t) - PQ \).

Then

\[ pdq - H dt = P dQ - K dt + \frac{\partial F_2}{\partial q} dq + \frac{\partial F_2}{\partial P} dP + \frac{\partial F_2}{\partial t} dt - P dQ - Q dP \]

gives a cancellation of the \( P dQ \) terms, has the effect of leaving only the independent differentials, \( dq, dP, dt \).

(This is purpose of \(-PQ\) term.)
Then:

\[ p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P} \]

\[ K = H + \frac{\partial F_2}{\partial t} \]

Now just summarize other case.

Case 3. \((p, Q)\) independent.

let \( F = F_3(p, Q) + pq \)

\[ q = -\frac{\partial F_3}{\partial p}, \quad P = -\frac{\partial F_3}{\partial Q} \]

\[ K = H + \frac{\partial F_3}{\partial t} \]

Case 4. \((p, P)\) independent.

let \( F = F_4(p, P) + pq - EP \)

\[ q = -\frac{\partial F_4}{\partial p}, \quad Q = \frac{\partial F_4}{\partial P} \]

\[ K = H + \frac{\partial F_4}{\partial t} \]

These are easy to remember, except for the signs.

This has been for 1 DOF. For \( >1 \) DOF, you can concoct generating for's which are of different types in the different DOF's. For example, in 2 DOF:

\[ F(q_1, p_2, P_1, P_2) \]

Type 2 in #1 DOF

Type 4 in #2 DOF.
Next let's examine the question of the uniqueness of these genus fn's. To be specific, work with $F_1$.

Suppose given canonical $(q,p) \rightarrow (Q,P)$, and some $F_1(q,Q,t)$ exists for it.

Any other $F_1(q,Q,t)$?

Well, we know, $\mathbf{p} = p(q,Q) = \frac{\partial F_1}{\partial q}$

$\mathbf{P} = P(q,Q) = -\frac{\partial F_1}{\partial Q}$

$\Rightarrow$ Gradient of $F_1$ in $(q,Q)$ space known.

Therefore, $F_1(q,Q) = \int_{(q_0,Q_0)}^{(q,Q)} \mathbf{p}(q,Q) dq - \mathbf{P}(q,Q) dQ$

This line integral is independent of the path, so depends only on endpoints. This shows that $F_1$ is determined to within a constant.

Therefore, any 2 $F_1$'s which give the same C.T. differ only by a constant. Same thing holds for other C.T.'s.

This allows us to write a relation connecting all the genus fn's.

$F_1(q,Q,t) = F_2(q,P,t) - DQ$ (An important fact, will call on later.)

$= F_3(p,Q,t) + pq$

$= F_4(p,P,t) + pq - DQ$. to within constants.
Now I'd like to show you the relation between CT's and LM's.

Recall, defn of LM involved

\[ \delta \vec{z}_1 \cdot J \cdot \delta \vec{z}_2 = 0. \]

The \( \vec{z}'s \) stood for \( q's \) and \( p's \).

\[ \vec{z} = \begin{pmatrix} q \\ p \end{pmatrix}. \]

Now suppose we have another set of canonical coords,

\[ \vec{y} = \begin{pmatrix} q' \\ p' \end{pmatrix}. \]

Then

\[ \delta \vec{z}_1 = \delta \vec{z}_1^{\text{y}}, \delta \vec{y}_1 = \delta \vec{y}_1^{\text{y}}. \]

\[ \delta \vec{z}_2 = \delta \vec{y}_2. \]

Therefore,

\[ 0 = \delta \vec{z}_1 \cdot J \cdot \delta \vec{z}_2 = \delta \vec{y}_1 \cdot \delta \vec{y}_2. \]

Converting displacements from \( \vec{z}'s \) to \( \vec{y}'s \) to rewrite in new coods.

So we see that the defn of a LM is the same in any system of canonical coordinates.

This allows us to immediately write down lots of LM's.
Consider the surface,

\[
\mathbb{Q} = \mathbb{Q}(\mathbf{q}, \mathbf{p}) = \text{anything}
\]

\[
\mathbb{P} = \mathbb{P}(\mathbf{q}, \mathbf{p}) = \text{const.}
\]

This is now obviously a LM, essentially a copy of the new momentum space. (Because \(SP = 0\)).

So we see that the space of const. momentum tends any CT to always an LM. (So are the spaces of const. \(Q\)).

Look at this geometrically in phase space.

\[\mathbb{P} = \mathbb{C}_0\]
\[\mathbb{P} = \mathbb{C}_1\]
\[\mathbb{P} = \mathbb{C}_2\]

So by varying the value of \(\mathbb{P}\), you get a family of LM's; in fact, an \(N\)-parameter family.

Such a thing is called a foliation (Lat. folium = "leaf").

A can. xfm. foliates phase space into an \(N\)-parameter family of \(N\)-dimensional Lagrangian manifolds, namely, surfaces \(\mathbb{P} = \text{const.}\).
Give an example. Consider 1D SHO,

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2. \]

Now look at CT to AA variables, \((q, p) \rightarrow (\theta, I)\)

\[ q = \sqrt{\frac{2I}{\omega}} \sin \theta \]
\[ p = \sqrt{2I\omega} \cos \theta \]

What are surfaces \(I = \text{const}^2\)

\[ H = \omega I \]

In this case, the foliation is just that given by the orbits.

Now let's turn the problem around. Suppose we've given a foliation of phase space into LM's. Can we find a CT corresponding to this?

May seem an abstract question at this point, so let me point out where this would occur in practice.

Given some multi-dim. \(H\), don't know how to solve. But one thing you can do is pick ic, integrate on computer.

Under certain circumstances, this orbit will trace out and densely fill in a LM.

Now pick different ic, integrate again, get new LM.
So here you get a foliation into LM's, but you don't have any CT. Q: Can you find one?

Answer is yes; here's how.

\[ p \]

\[ \hat{\alpha} \]

\[ \hat{\alpha}_1 \]

\[ \hat{\alpha}_2 \]

\[ \hat{\alpha}_3 \]

First, arbitrarily label LM's with a set of parameters.

Call them \( \hat{\alpha} \), so as not to prejudice things.

\( \hat{\alpha} \) must be N-vector.

Now suppose that on each of these LM's, \( \bar{p} \) can be expressed as a fn. of \( \hat{\alpha} \). So, know

\[ \bar{p} = \bar{p}(\hat{\alpha}, \hat{\alpha}) \]  \hspace{1cm} (Different fn. for each LM.)

Then, since there are LM's, we know \( \bar{p} \) is a perfect gradient.

Therefore, set

\[ S(\hat{q}, \hat{\alpha}) = \int_{\hat{q}_0}^{\hat{q}} \bar{p}(\hat{q}, \hat{\alpha}) \cdot d\hat{q} \]

Finally, we simply interpret the \( \hat{\alpha} \)'s as new momenta.

Set \[ F_2(\hat{q}, \hat{E}) = S(\hat{q}, \hat{E}) \], and we have our CT.

You could also interpret \( \hat{\alpha} \)'s as new \( \alpha \)'s, use \( F_1 \).
Now I'd like to comment on the philosophy adopted in GM in regard to CT's and the HJ eqn.

Go back to $F_1(q, Q, t)$. Given some $H(q, p, t)$ to solve.

$$K(q, p, t) = H(q, p, t) + \frac{\partial F_1}{\partial t}(q, Q, t).$$

Idea is, choose $F_1$ so new $K$ is easier to solve than old one. Well, what would be a $K$ that would be easy to solve?

Think big. Take $K = 0$. Then $Q = 0 \Rightarrow$ new motion is trivial.

Find $F_1$ such that

$$H(q, \frac{\partial F_1}{\partial q}(q, Q, t), t) + \frac{\partial F_1}{\partial t}(q, Q, t) = 0.$$  

This obviously is the HJ eqn; idea is, solve this, get $F_1$, get your CT.

But notice that this version of the HJ eqn. is different from what we had in WKB theory, because there we didn't have any $Q$'s.

What this means is, t-dep WKB theory requires only a particular soln, $S(q,t)$;

Here we need an N-parameter soln, $F_1(q, Q, t)$, called a complete soln (Jacobi). Not same as general soln.
But we already know about one complete soln; this is Hamilton's principal fn.

\[ \mathcal{S}_H(q_0, t_0, q, t). \]

Note, \( \frac{\partial \mathcal{S}_H}{\partial q} = \text{final } p \)

\[ \frac{\partial \mathcal{S}_H}{\partial t} = -\text{final } H = -H(q, p, t) \]

\[ = -H(q, \frac{\partial \mathcal{S}_H}{\partial q}, t). \]

Furthermore, \( \mathcal{S}_H \) is an \( N \)-parameter soln; we can take the \( q_0 \)'s as the parameters.

Identify \( q_0 = Q \), set

\[ F_i(q, Q, t) = \mathcal{S}_H(q_0, t_0, q, t). \]

Now, what CT does this generate? Well, set

\[ I = -\frac{\partial F_i}{\partial Q} = -\frac{\partial \mathcal{S}_H}{\partial q_0} = p_0. \]

So we see,

\( (q, p) = \text{old coords = final } (q, p) \)

\( (Q, p) = \text{new coords = init. } (q_0, p_0) \).

to the CT generated by Ham's princ. fn. is the CT which undoes the time evol; it is

\[ (q, p) \rightarrow (q_0, p_0). \]

And the latter are constants, of course.
Now we turn to a new subject, which is the stationary phase approx.

First, let me motivate this within WKB theory. Suppose we want to do WKB theory in momentum space. So, given an $x$-space WKB wave for, find what the $\hat{p}$ space for is. Well, one way to do this is to do the FT.

$$
\psi(\hat{p},t) = \int \frac{d^nx}{(2\pi)^{3/2}} e^{-i\hat{p} \cdot \vec{x}/\hbar} A(x) e^{\frac{i}{\hbar} S(x,t)}
$$

(Just look at one branch.) In general, you can't do this integral exactly. But you can do it approximately, by the SPA, and it turns out that this approx. is equiv. to a small $\hbar$ approx., and therefore equiv. to WKB approx. Since it's not exact anyway, using an approx. of the same order to evaluate $\psi$ is completely consistent.

Note that if $\hbar$ is small, integrand has rapid oscillations, and that's the essential ingredient in the SPA.
$$H(x, p, t) = \text{Quant. Ham.}$$

$$H\left(\frac{i\hbar}{2p}, p, t\right) \varphi(p, t) = i\hbar \frac{\partial \varphi}{\partial t}(p, t)$$

$$\varphi(p, t) = \mathcal{B}(p, t) e^{\frac{i}{\hbar} \mathcal{T}(p, t)}.$$ 

$$\varphi(p_0, t) = \int_{-\infty}^{+\infty} dx \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} A(x, t) e^{\frac{i}{\hbar} S(x, t)}.$$ 

Stationary Phase Method
Steep Descent
Saddle Point
Mountain Pass

$$\int_{c} \frac{f(z)}{dz} e^{\int_{c} \frac{f(z)}{dz} dz}$$

$$z = x + iy$$
\[
\begin{align*}
(u_{xx} \quad u_{xy})(W_x)_x &= \lambda (W_x)_x^H \\
(u_{xy} \quad u_{yy})(W_y)_y &= \lambda (W_y)_y^H \\
\text{can relate to 2nd derivs of } f.
\end{align*}
\]

\[f(z) = u_x + i v_x = u_x - i u_y\]
\[\Rightarrow (f^n(z)) = u_{xx} - i u_{xy}\]
\[u_{xx} W_x + u_{xy} W_y = \lambda W_x\]
\[u_{xy} W_x - u_{xx} W_y = \lambda W_y\]

\[f''(z) W = \begin{pmatrix} (u_{xx} - i u_{xy}) & (u_{xx} + i u_{xy}) \\ -u_{xy} & -u_{xy} \end{pmatrix} W \]
\[= \begin{pmatrix} \lambda W_x + i \lambda W_y \\ -u_{xy} W_x - u_{xx} W_y \end{pmatrix}
\]
\[= \lambda W_x - i \lambda W_y = \lambda W^*\]

\[f''(z) W = \lambda W^*\]

\[f''(z) W_+ = \lambda W_+^*\]
\[f''(z) W_- = -\lambda W_-^*\]

\[W_+ = \pm \sqrt{\frac{\lambda}{f''(z)}}\]
\[W_- = \pm \sqrt{-\frac{\lambda}{-f''(z)}}\]