Announce Office Hours.

F 12-1, (but not this Fri.).
W 12-1 this week.

Physics 205B Advanced Dynamics.

In the past, Physics 205B has been used for a variety of subjects which can be considered to fall under "Advanced Dynamics," and this semester the topic will be WKB Theory and Semiclassical mech.

One thing 205B is supposed to do is to present some material in current research topics, and this semester will be no exception. 1st half: older, 2nd half: research.

WKB theory can be applied to any wave eqn., but we will work mainly with the NR Sch. eqn.

Prerequisites:

205A: H-J eqn., gen. fur., chaos, KAM thms., invariant tori, etc.

221A: Grad. stud., perspective on QM.

Level of math: Nothing special.

Text: None

Handouts: 500 pages of reprint #18 (Copymat).
(Some reading assignments)
We will work with NR Sch. eqns,

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi = i \hbar \frac{\partial \psi}{\partial t} \]

This is obviously a kind of amplitude-phase decomposition of the wave function and when it is small, it's supposed to invoke a mental image of a slowly modulated wave train, or a wave train with a rapidly varying phase.

WKB theory begins with the

\[ \psi(x, t) = A(x, t) e^{i S(x, t)} \]

"Eikonal" mostly used in optics

"WKB" in QM. ( Wentzel, Kramers, Brillouin).
So if you want to know where this ansatz comes from, it comes from thinking about a picture like this.

In a mathematical sense, this ansatz constitutes a kind of small $\hbar$ expansion of the wave $\psi$. Notice that $\psi$ is not represented as a power series in $\hbar$; that wouldn't work, because as $\hbar \to 0$, $\psi$ doesn't really approach any limit at all. Instead, it has an essential singularity as $\hbar \to 0$.

$$(\psi = \psi_0 + \hbar \psi_1 + \ldots \text{ won't work})$$

The WKB ansatz is more like an expansion in $\hbar$ of the logarithm of $\psi$. To see this, rewrite the WKB ansatz,

$$\psi(x,t) = e^{\frac{i}{\hbar} \left[ S(x,t) + \frac{\hbar}{2} \ln A(x,t) \right]}$$

The exponent is beginning to look like a power series, so let's extend it all the way out, and write something like this:

$$\psi(x,t) = e^{\frac{i}{\hbar} W(x,t)}$$

$$W(x,t) = W_0(x,t) + \left( \frac{\hbar}{i} \right) W_1(x,t) + \left( \frac{\hbar^2}{i^2} \right) W_2 + \ldots$$

$$S(x,t) \quad \ln A(x,t) \quad \hbar \text{ o.t.}$$

So usual WKB ansatz is only first 2 terms; often we only three, because most important.
Might mention, this power series usually turns out not to be convergent, but only asymptotic.

So let's begin our expansion by transforming the Sch. Eqn. from \( \psi \) to the log of \( \psi \), to get an Eqn. for \( W \).

\[
\psi = e^{-iW/\hbar}
\]

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} = -\frac{\partial W}{\partial t} \psi
\]

\[
\nabla \psi = \frac{1}{\hbar} \nabla W \psi
\]

\[
\nabla^2 \psi = \left[ \frac{i}{\hbar} \nabla^2 W - \frac{1}{\hbar^2} (\nabla W)^2 \right] \psi
\]

\[
-\frac{i\hbar}{2m} \nabla^2 W + \frac{1}{2m} (\nabla W)^2 + V = -\frac{\partial W}{\partial t}
\]

Exact rep. of Sch. Eqn., at least if you can ignore the fact that the Lin. fn. has multiple branches. But nonlinear SC mech. typically replaces a linear problem by a nonlinear one.

Now expand \( W \). Lowest order is easy:

\[
\Theta(\xi_0): \quad \frac{1}{2m} (\nabla S)^2 + V(x,t) + \frac{\partial S}{\partial t} = 0
\]
Can write in terms of classical Hamiltonian,

\[ H(\vec{x}, \vec{p}, t) = \text{class. Ham.} \]

\[ H(\vec{x}, \vec{p}, t) + \frac{\partial S}{\partial t} = 0 \]

Hamilton-Jacobi eqn. of class. mech.

Interesting fact:
It was shown by Hamilton that if you can find a sufficiently large class of solns of this eqn, it is equivalent to finding the complete soln of the classical eqns of motion.

Now go on to 1st order, get eqn for \( W_1 \),

\[ -\frac{i\hbar}{2m} \nabla^2 S + \frac{1}{m} \nabla S \cdot \nabla (\frac{k}{i} W_1) = -\frac{1}{\hbar} \frac{\partial W_1}{\partial t} \]

cancel \( i, \hbar \), set \( W_1 = \ln A \),

\[ \frac{1}{2m} \nabla^2 S + \frac{1}{m} \nabla S \cdot \nabla \ln A + \frac{\partial}{\partial t} \ln A = 0 \]

Now write \( \nabla \ln A = \frac{1}{A} \nabla A \),

\[ \frac{\partial}{\partial t} \ln A = \frac{1}{A} \frac{\partial A}{\partial t} \]

mult this by \( A \),

\[ \frac{1}{2m} A \nabla^2 S + \frac{1}{m} \nabla S \cdot \nabla A + \frac{\partial A}{\partial t} = 0 \]
Called amplitude transport eqn.

The idea is, first solve HJ eqn. for $S$;
then plug soln for $S$ into AT eqn., solve for $A$.

So what we see developing here is a hierarchy of eqns: If you carry this to higher order, you will find at each order an eqn. for $W_n$, which involves the soln of all the previous eqns. So you solve them one at a time.

The AT eqn can be cast into a more recognizable form, if we make a simple substitutions. Final, act

$$\vec{v}(\vec{x}, t) = \frac{1}{m} \nabla S(\vec{x}, t).$$

The notation is supposed to suggest that $\vec{v}$ is a velocity (in fact, a velocity field). And if you're up on your CM, you know that $S$, the soln of the HJ eqn., is a generating fn. of a ct, and that $\nabla S$ therefore is a momentum,

$$\vec{p}(\vec{x}, t) = \nabla S(\vec{x}, t),$$

so this is just the relation betw. $\vec{v}$ and $\vec{p}$ for the Hamiltonian we have chosen.

If this connection with gen. fn.'s is not immediately obvious to you, don't worry, we'll discuss it in more detail later.

Anyway, the second step is to set

$$\rho(\vec{x}, t) = A(\vec{x}, t)^2;$$

notation supposed to suggest a density. Then the AT eqn becomes,
\[ \frac{1}{2} \nabla \cdot \vec{A} + \nabla \cdot \vec{V} + A \frac{\partial A}{\partial t} = 0 \]

or
\[ \rho \nabla \cdot \vec{V} + \vec{V} \cdot \nabla \rho + \frac{\partial \rho}{\partial t} = 0 \]

or
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{V} \rho) = 0 \]

and the 1st eqn. becomes a continuity eqn. for a conserved quantity, whose density is \( \rho \).

So far this has been rather formal, so let me now introduce some physical interpretation into it.

Go back to picture of wave front. To say that \( A \) is slowly varying means that if you look at \( \vec{V} \) on the scale of a few wavelengths, then \( A \) doesn't change very much.

In other words, it looks like a local plane wave. This plane wave can be associated with a local wave vector and momentum, by the de Broglie relations.

\[ \vec{k} = \frac{2\pi}{\lambda} \]
\[ \vec{p} = h \vec{k} = \frac{2\pi h}{\lambda} \]

If you look at a far distant point, then \( A \) may be substantially different, and so may \( A \). But if the wavelength is small in comparison to the scale of variation of \( A \), then it's reasonable to think of \( A = A(x, t) \), and therefore
Do a similar calculation, but in time. Here we have, 
(Hold \(x\) fixed, wait for time \(T\) = period of wave at 
fixed \(x\), such that wave goes thru \(2\pi\) of phase).

\[
\frac{1}{\hbar} \frac{S(x, t+T)}{T} = \frac{1}{\hbar} S(x, t) - 2\pi.
\]

(Minus sign is a convention; think \(e^{-i(kx - \omega t)}\) for 
plane wave.)

Then we have

\[
\frac{1}{\hbar} T \frac{2S}{3t} = -2\pi
\]

\[
\frac{\partial S}{\partial t} = -\frac{2\pi \hbar}{T} = -k\omega = -E. 
\text{(Einstein)}
\]

So, we get also an energy field,

\[
E(x, t) = -\frac{\partial S}{\partial t} (x, t)
\]

So we seem to have a way of associating, at each space-time 
point, an energy and momentum with the given wave.

Furthermore, look at what HJ eqn. tells us:

\[
H_{cl} (\hat{x}, \nabla S, t) + \frac{\partial S}{\partial t} = 0 
\text{ (remember, came \text{from Schr. eqn.})}
\]

This says, \(H (\hat{x}, \hat{p}(\hat{x}, t), t) = E(x, t)\).
\( \vec{p} = \vec{p}(\vec{x}, t) \).

So this WKB type wave function seems to be naturally associated with a momentum field.

Now let's relate \( \vec{p} \) to the phase \( S \) of the WKB wave function. For simplicity, work in 1D. Consider 2 points, \( x \) and \( x + \lambda \).

The phase must increment by \( 2\pi \) in this interval, so we must have

\[
\frac{1}{\hbar} S(x + \lambda, t) = \frac{1}{\hbar} S(x, t) + 2\pi.
\]

Expand to 1st order in \( \lambda \),

\[ \frac{1}{\hbar} \lambda \frac{\partial S}{\partial x} = 2\pi, \]

\[ \frac{\partial S}{\partial x} = \frac{2\pi \hbar}{\lambda} = \vec{p} \text{ (de Broglie).} \]

Generalize to 3D,

\[ \vec{p}(\vec{x}, t) = \nabla S(\vec{x}, t). \]
i.e. the $\vec{p}$ and $E$ fields are not independent, but are related to one another by the classical $H$.

So what this is telling us is that there is an interpretation of the $\vec{p}$ in terms of, not one particle, but a swarm of particles, one for each $x$ point, with momentum $\vec{p}(x,t)$ and energy $E(x,t)$. Think:

can get $\vec{p}, E$ from $\rho(x,t)$.

Not all the particles have to have the same energy. Furthermore, as we will show, it turns out that the dynamics dictated by the $H_S$ equation is equivalent to letting each of these particles move under the classical $H$, each moving to new places and acquiring new $\vec{p}$ and $E$ in the process.

So, you have to distinguish $\vec{p}$ and $E$ as seen at one position at a fixed time, and $\vec{p}$ and $E$ as seen along a particle orbit. Of course, if $\partial H/\partial t = 0$, then $E$ is constant along a particle orbit, but not necessarily fixed at a fixed $x$, because as time progresses, you will be looking at new particles (nothing says they all have to have the same energy).

This picture I have described is basically the physical picture which goes along with the solution of the $t$-dependent HJ equation (not found in Goldstein).

Now, as the particles move around, they will generally either bunch up, or stretch out, depending on gradients and time derivatives of the velocity field. But the $\rho$ doesn't say anything about the density of particles. That is given by $\Lambda$. 