Physics 139: Problem Set 13 solutions

May 2, 2014

Hartle 21.8

Calculate $R_{\hat{\tau}\hat{r}\hat{\tau}\hat{r}}$ for the Schwarzschild metric in the frame of the freely falling observer in (20.81). To do this first calculate $R_{\alpha\beta\gamma\delta}$ in the Schwarzschild coordinate basis, and then use (21.24) to get the component in the freely falling frame. Does your answer agree with (21.30a)?

$$\begin{aligned} R_{\hat{\tau}\hat{r}\hat{\tau}\hat{r}} &= g_{\alpha\sigma} R^{\sigma}{}_{\beta\gamma\delta} (e_{\hat{\tau}})^{\alpha} (e_{\hat{r}})^{\beta} (e_{\hat{\tau}})^{\gamma} (e_{\hat{r}})^{\delta} \\ &= \left(g_{tt} R^{t}{}_{\beta\gamma\delta} (e_{\hat{\tau}})^{t} + g_{rr} R^{r}{}_{\beta\gamma\delta} (e_{\hat{\tau}})^{r} \right) (e_{\hat{r}})^{\beta} (e_{\hat{\tau}})^{\gamma} (e_{\hat{r}})^{\delta} \end{aligned}$$

Since $e_{\hat{\tau}}$ and $e_{\hat{r}}$ only have components in the r and t coordinate directions, we need only consider the following components of the Riemann tensor

$$R^{t}{}_{rtr} = \frac{2M}{r^2(r-2M)}, \;\; R^{r}{}_{trt} = \frac{2M(2M-r)}{r^4}$$

(the other necessary components are related to these by antisymmetry of the last two indices). Plugging into the above using

$$(e_{\hat{\tau}})^{\alpha} = \left\{ (1-2M/r)^{-1}, -(2M/r)^{1/2}, 0, 0 \right\}, \;\; (e_{\hat{r}})^{\alpha} = \left\{ -(2M/r)^{1/2}(1-2M/r)^{-1}, 1, 0, 0 \right\}$$

we recover (21.30a)

$$R_{\hat{\tau}\hat{r}\hat{\tau}\hat{r}} = -\frac{2M}{r^3}$$

Hartle 21.11

Problem 8.12 introduced the two-dimensional hyperbolic plane and claimed it has constant negative curvature. Does it? Calculate $R \equiv R^{\alpha}_{\alpha}$ for this two-dimensional geometry to find out.

The metric for the two-dimensional hyperbolic plane is

$$ds^2 = y^{-2}\left(dx^2 + dy^2\right), \;\; y \geq 0$$

The non-zero Christoffel symbols are

$$\Gamma^{x}_{xy} = -\Gamma^{y}_{xx} = \Gamma^{y}_{yy} = -1/y$$

Since the metric is diagonal, we only need the diagonal components of the Ricci tensor

$$R = y^2 \left(R_{xx} + R_{yy}\right) = y^2 \left(R^k{}_{xkx} + R^k{}_{yky}\right) = y^2 \left(R^y{}_{xyx} + R^x{}_{yxy}\right)$$

$$\begin{aligned} R^y{}_{xyx} &= \Gamma^y_{xx,y} - \Gamma^y_{yx,x} + \Gamma^y_{yi}\Gamma^i_{xx} - \Gamma^y_{xi}\Gamma^i_{yx} \\ &= \Gamma^y_{xx,y} + \Gamma^y_{yy}\Gamma^y_{xx} - \Gamma^y_{xx}\Gamma^x_{yx} \\ &= -\frac{1}{y^2} - \frac{1}{y^2} + \frac{1}{y^2} \\ &= -y^{-2} \end{aligned}$$

and

$$\begin{aligned} R^x{}_{yxy} &= \Gamma^x_{yy,x} - \Gamma^x_{xy,y} + \Gamma^x_{xi}\Gamma^i_{yy} - \Gamma^x_{yi}\Gamma^i_{xy} \\ &= -\Gamma^x_{xy,y} + \Gamma^x_{xy}\Gamma^y_{yy} - \Gamma^x_{yx}\Gamma^x_{xy} \\ &= -\frac{1}{y^2} + \frac{1}{y^2} - \frac{1}{y^2} \\ &= -y^{-2} \end{aligned}$$

Then

$$R = y^2 \left(-y^{-2} - y^{-2}\right) = -2$$

so that the two-dimensional hyperbolic plane does, in fact, have constant curvature.

Hartle 21.12

a) *For the wormhole metric (7.39), calculate the components of the Riemann curvature in an orthonormal basis whose vectors point along the* (t, r, θ, ϕ) *coordinate axes.*

b) *Show that a stationary observer at the wormhole throat feels no tidal gravitational forces.*

c) *Show that an observer moving radially through the throat with speed* V *as measured by a stationary observer at any point along its trajectory, experiences tidal gravitational forces proportional to* V^2.

d) *How do these tidal forces depend on the radius of the throat? What combination of* b *and* V *would make for a survivable trip through the wormhole?*

a)

The components of the Riemann curvature in an orthonormal basis pointing along the (t, r, θ, ϕ) coordinate directions are found in the same way as in the previous problem in the Schwarzschild geometry. The non-vanishing components are

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = -R_{\hat r\hat\theta\hat r\hat\theta} = -R_{\hat r\hat\phi\hat r\hat\phi} = \frac{b^2}{(r^2 + b^2)^2}$$

in which we have used the components of the orthonormal basis in the coordinate directions

$$\begin{aligned}
(e_{\hat{t}})^\alpha &= \{1,0,0,0\} \\
(e_{\hat{r}})^\alpha &= \{0,1,0,0\} \\
(e_{\hat{\theta}})^\alpha &= \left\{0,0,(b^2+r^2)^{-1/2},0\right\} \\
(e_{\hat{\phi}})^\alpha &= \left\{0,0,0,(b^2+r^2)^{-1/2}(\sin\theta)^{-1}\right\}
\end{aligned}$$

b)

Stationary observers have only t-components of their four-velocities. But there are no t-components of the Riemann tensor. The right hand side of the equation of geodesic deviation therefore vanishes.

c)

The four-velocity of an observer falling radially is given by (8.21). Expressed in terms of the velocity V measured by a stationary observer, this is

$$u^{\hat{\alpha}} = \{\gamma, \gamma V, 0, 0\}$$

where $\gamma = (1-V^2)^{-1/2}$. (A simple way to see this is to note that the (t,r) part of the metric is like flat spacetime.) Then, for instance,

$$(\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\boldsymbol{\chi})^{\hat{\theta}} = -R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}}\left(u^{\hat{r}}\right)^2\chi^{\hat{\theta}} = (\gamma V)^2\,\frac{b^2}{(r^2+b^2)^2}\chi^{\hat{\theta}}$$

proportional to V^2 as advertised.

d)

The tidal forces are largest at the smallest r which is 0. To survive traversing the wormhole $(V^2/b^2)\times$ (length of the observer)$\times$(mass of the observer) must be less than the maximum tolerable force. For maximum survivability it's best to go through a large wormhole, and slowly.

Hartle 21.15

Calculate R_{AB} for the metric on the sphere

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Work through this problem by hand to make sure you understand how to do it without the computer program.

First calculate the non-zero Christoffel symbols.

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,\ \Gamma^\phi_{\phi\theta} = \cot\theta$$

The Ricci curvature components are

$$R_{AB} = R^C{}_{ACB}$$

In particular,

$$\begin{aligned} R_{\theta\theta} = R^{\phi}{}_{\theta\phi\theta} &= -\Gamma^{\phi}_{\phi\theta,\theta} + \Gamma^{\phi}_{\phi C}\Gamma^{C}_{\theta\theta} - \Gamma^{\phi}_{\theta C}\Gamma^{C}_{\theta\phi} \\ &= -\Gamma^{\phi}_{\phi\theta,\theta} - \Gamma^{\phi}_{\theta\phi}\Gamma^{\phi}_{\theta\phi} \\ &= \csc^2\theta - \cot^2\theta \\ &= 1 \end{aligned}$$

Continuing on,

$$R_{\theta\phi} = R^{C}{}_{\theta C\phi} = g^{AC}R_{A\theta C\phi} = 0$$

because the inverse metric is diagonal and each pair of indices of the Riemann tensor are anti-symmetric. Finally,

$$\begin{aligned} R_{\phi\phi} = R^{\theta}{}_{\phi\theta\phi} &= \Gamma^{\theta}_{\phi\phi,\theta} - \Gamma^{\theta}_{\theta\phi,\phi} + \Gamma^{\theta}_{\theta A}\Gamma^{A}_{\phi\phi} - \Gamma^{\theta}_{\phi A}\Gamma^{A}_{\phi\theta} \\ &= \Gamma^{\theta}_{\phi\phi,\theta} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} \\ &= \sin^2\theta - \cos^2\theta + \cos^2\theta \\ &= \sin^2\theta \end{aligned}$$

Hartle 21.17

(The Schwarzschild geometry satisfies the Einstein equations.) Insert the Christoffel symbols for the Schwarzschild geometry into (21.33) and evaluate. You should find that $R_{\alpha\beta} = 0$ identically for each of the ten possible combinations of α and β, thus proving that the Scharzschild geometry is a solution of the empty space Einstein equations.

Just do the algebra. Answer is in the question.